Math 42 Autumn 2004 Homework 4 Solutions

6.3 #1 \( y = 2 - 3x \) so

\[
L = \int_{-2}^{1} \sqrt{1 + (-3)^2} \, dx = 3\sqrt{10}.
\]

By the distance formula,

\[
L = \sqrt{[1 - (-2)]^2 + [(1 - 8)^2]} = 3\sqrt{10}.
\]

6.3 #6 \( \frac{dx}{dt} = e^t - e^{-t} \) and \( \frac{dy}{dt} = -2 \), so

\[
(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2.
\]

and since the expression in parentheses is positive,

\[
L = \int_0^3 (e^t + e^{-t}) \, dt = [e^t - e^{-t}]_0^3 = e^3 - e^{-3}.
\]

6.3 #8 \( y' = \frac{x^2}{2} - \frac{x^{-2}}{2} \), so

\[
1 + (y')^2 = 1 + \frac{x^4}{4} - \frac{1}{2} + \frac{x^{-4}}{4} = \frac{x^4}{4} + \frac{1}{2} + \frac{x^{-4}}{4} = \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2,
\]

and since the expression in parentheses is positive,

\[
L = \int_{1/2}^{1} \left(\frac{x^2}{2} + \frac{x^{-2}}{2}\right)^2 \, dx = \left[\frac{x^3}{6} - \frac{1}{2}\right]_{1/2}^{1} = \frac{31}{48}.
\]

6.3 #10 With a little algebra and the Pythagorean identity,

\[
\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (a\theta)^2,
\]

so

\[
L = \int_{0}^{\pi} a\theta \, d\theta = a \left[\frac{1}{2}\theta^2\right]_{0}^{\pi} = \frac{1}{2}a\pi^2.
\]

6.3 #12 \( y' = \sec^2 x \), so \( L = \int_{0}^{\pi/4} \sqrt{1 + \sec^2 x} \, dx \). We apply Simpson’s Rule with \( a = 0, b = \pi/4, n = 10, \Delta x = \pi/40, \) and \( f(x) = \sqrt{1 + \sec^2 x} \). So

\[
L \approx \frac{\pi}{120} [f(0) + 4f(\pi/40) + 2f(2\pi/40) + \cdots + 4f(9\pi/40) + f(\pi/4)] \approx 1.278.
\]

6.3 #22 \( y' = -\frac{1}{\sqrt{20}}(x - 50) \), so

\[
L = \int_{0}^{80} \sqrt{1 + \frac{1}{20^2}(x - 50)^2} \, dx = 20 \int_{-5/2}^{3/2} \sqrt{1 + u^2} \, du,
\]

using the substitution \( u = \frac{1}{\sqrt{20}}(x - 50) \). By Formula 21 in the table of integrals, this is

\[
L = 10\left[u\sqrt{1 + u^2} + \ln(u + \sqrt{1 + u^2})\right]_{-5/2}^{3/2} = \frac{15}{2}\sqrt{13} + \frac{25}{2}\sqrt{29} + 10\ln\left(\frac{3 + \sqrt{13}}{-5 + \sqrt{29}}\right).
\]
6.4 #4 With the substitution \( u = 1 + r \),
\[
h_{\text{ave}} = \frac{1}{6 - 1} \int_1^6 \frac{3}{(1 + r)^2} dr = \frac{3}{5} \int_2^7 u^{-2} du = -\frac{3}{5} \left[ u^{-1} \right]_2^7 = \frac{3}{14}.
\]

6.4 #6 (a) Integrating by parts,
\[
\int f(x) dx = -\frac{1}{3} \ln x + \frac{1}{2} \left[ x \ln x - x \right]_1^3 = \frac{3}{2} \ln 3 - 1.
\]

(b) \( f_{\text{ave}} = f(c) \) iff \( \frac{1}{2} \ln 3 - 1 = \ln c \). Solving for \( c \) we obtain \( c = \frac{3\sqrt{3}}{e} \).

6.4 #9 \( f \) is continuous on \([1, 3]\), so by the Mean Value Theorem for Integrals there exists a number \( c \) in \([1, 3]\) such that \( \int_1^3 f(x) dx = f(c)(3 - 1) \). So for that \( c, f(c) = \frac{1}{2} \int_1^3 f(x) dx = 4 \).

6.4 #12 (a) \( v_{\text{ave}} = \frac{1}{12 - 0} \int_0^1 2v(t) dt = \frac{1}{12} I \). Using (for example) Simpson’s rule with \( n = 6 \) and \( \Delta t = 2 \) to estimate \( I \).
\[
I \approx \frac{2}{3} (0 + 4(21) + 2(38) + 4(50) + 2(60) + 4(66) + 70) = \frac{1628}{3}.
\]
Thus, \( v_{\text{ave}} \approx \frac{1}{12} \cdot \frac{1628}{3} = \frac{407}{9} \approx 45 \text{ km/h} \).

(b) Estimating from the graph, \( v(t) = 45 \) when \( t \approx 5 \text{ s} \).

6.5 #4 If \( k \) is the constant for this spring, then \( 25 = k(0.3 - 0.2) = (0.1)k \) (since 10 cm is 0.1 m), so \( k = 250 \text{ N/m} \) and \( f(x) = 250x \). So
\[
W = \int_0^{0.05} 250x \, dx = [125x^2]_0^{0.05} = 125(0.0025) = 0.3125 \text{ J}.
\]

6.5 #14 Set up coordinates with the origin at the center of the circle at the top of the hemisphere. We divide up the water in the tank into horizontal circular slices, since each such slice will be raised the same distance. At a given value of \( y \) such that \(-5 \leq y \leq 0\), the water will be raised a distance \((-y)\). The slice at that height has radius \( \sqrt{25 - y^2} \), and so its volume is about \( \pi(25 - y^2) \Delta y \). So the work to pump out the thin slice at that height is
\[
\Delta W = (62.5)\pi(25 - y^2)(-y) \Delta y \text{ ft.-lbs}.
\]
Therefore the total work is
\[
W = \int_{-5}^{0} (62.5)\pi(25 - y^2)(-y) \, dy = (62.5)\pi \left[ -\frac{25}{2} y^2 + \frac{1}{4} y^4 \right]_{-5}^{0} = (62.5)\pi \frac{625}{4} \approx 3.074 \text{ ft.-lbs}.
\]

6.5 #15 Method 1: We chop up the interval \([V_1, V_2]\) into small subintervals, which corresponds to chopping up the space between the two positions of the piston head into thin circular discs of volume \( \Delta V \). The thickness of one of these discs is \( \frac{\Delta V}{\pi r^2} \), which is the distance the piston head is moved by the little change \( \Delta V \) in the volume of the gas. The force exerted on the piston head to move it that distance is about \( F \approx \pi r^2 P(V) \) (since the volume is changing only a little bit, the pressure, and hence the force, at the beginning and end of that little bit of change should be almost the same). So the little bit of work exerted to make that small change in volume is about
\[
\Delta W \approx \pi r^2 P(V) \frac{\Delta V}{\pi r^2} = P(V) \Delta V.
\]
So the total work done by the expanding gas is
\[
W = \int_{V_1}^{V_2} P(V) \, dV.
\]
Method 2: $V = \pi r^2 x$, so $V$ is a function of $x$ and $P$ can also be regarded as a function of $x$. If $V_1 = \pi r^2 x_1$ and $V_2 = \pi r^2 x_2$, then

$$W = \int_{x_1}^{x_2} F(x)dx = \int_{x_1}^{x_2} \pi r^2 P(V(x))dx = \int_{x_1}^{x_2} P(V(x))dV = \int_{V_1}^{V_2} P(V)dV$$

by the Substitution Rule.

6.5 #20 Set up coordinates so that the origin is at the center of the top of the semicircular dam. We divide up the interval $[-10, -5]$ on the $y$-axis (since that is where the water is) into $n$ subintervals of length $\Delta y = \frac{5}{n}$. The water against the dam at heights corresponding to the $i$th subinterval is at a depth of about $-5 - y^*_i$, where $y^*_i$ is any number in the $i$th subinterval. Therefore the pressure being exerted by the water at those heights is about $1000g(-5 - y^*_i)$. The surface area of the strip of the dam corresponding to that subinterval is about $2\sqrt{100 - (y^*_i)^2}\Delta y$ (approximating the strip by a rectangle). So the force on the $i$th strip is about $2000g\sqrt{100 - (y^*_i)^2}(-5 - y^*_i)\Delta y$, and the total force is about

$$F \approx \sum_{i=1}^{n} 2000g\sqrt{100 - (y^*_i)^2}(-5 - y^*_i)\Delta y.$$

Taking the limit as $n \to \infty$ and substituting $u = -y$,

$$F = \int_{-5}^{-10} 2000g\sqrt{100 - y^2}(-5 - y)dy = 2000g\left(\int_5^{10} u\sqrt{100 - u^2}du - 5\int_5^{10} \sqrt{100 - u^2}du\right).$$

The first integral can be evaluated with a further substitution $v = 100 - u^2$; the second can be evaluated using Formula 30 in the table of integrals. We end up with

$$F = (250,000)g\left(\frac{3\sqrt{3}}{2} - \frac{2\pi}{3}\right) \approx 1.23 \times 10^6 \text{ N}.$$

6.5 #24 Set up coordinates with the origin at the center of the bottom of the semicircular gate. We divide up the dam into thin horizontal strips, since the depth of the water (and therefore the pressure) is about constant along each such strip. The depth of the water at height $y$ is $10 - y$, and so the pressure there is $1000g(10 - y)$. The area of one of the thin horizontal strips is about $2\sqrt{4 - y^2}\Delta y$, and so the force on one of the strips is about $2000g\sqrt{4 - y^2}(10 - y)$. Therefore the total force against the gate is

$$F = 2000g\int_0^2 (10 - y)\sqrt{4 - y^2}dy = 2000g\left(10\int_0^2 \sqrt{4 - y^2}dy - \int_0^2 y\sqrt{4 - y^2}dy\right).$$

The first integral can be evaluated with the substitution $u = 4 - y^2$; the second can be evaluated using Formula 30 in the table of integrals, or by noticing that $\int_0^2 \sqrt{4 - y^2}dy$ is the area of a semicircle with radius 2. We end up with

$$F = 1000g\left(20\pi - \frac{16}{3}\right) \approx 5.635 \text{ N}.$$

6.6 #12 By the Total Change Theorem, the increase is

$$\int_5^9 (2200 + 10e^{0.8t})dt = \left[2200t + \frac{10e^{0.8t}}{0.8}\right]_5^9 = 8800 + 12.5(e^{7.2} - e^4) \approx 24,860.$$

6.6 #15

$$\int_0^{12} c(t)dt = \int_0^{12} \frac{1}{4}t(12 - t)dt = \frac{1}{4}\left[6t^2 - \frac{1}{3}t^3\right]_0^{12} = 72 \text{ mg s/L}.$$

So the cardiac output is $F = \frac{A}{L_s} c(t)dt = \frac{8}{72} = \frac{1}{9} \text{ L/s}$. 

6.6 #16 Using Simpson’s Rule with $\Delta t = 2$,

$$\int_0^{20} c(t) \, dt \approx \frac{2}{3} [0 + 4(2.4) + 2(5.1) + \cdots + 4(0.7) + 0] = \frac{2}{3} (110.8) \text{ mg/s/L.}$$

So $F \approx \frac{34}{221.6} \approx 0.1083 \text{ L/s.}$