Pg. 28, 1:  

If  

\[ f(x, y) = u(x, y) + iv(x, y) \]  

and  

\[ g(r, s) = \alpha(r, s) + i\beta(r, s), \]  

then  

\[ g(f(x, y)) = \alpha(u(x, y), v(x, y)) + i\beta(u(x, y), v(x, y)). \]  

Using the chain rule, we have  

\[ (\text{Re } g(f))_x = \alpha_r(u, v)u_x + \alpha_s(u, v)v_x \]  

\[ (\text{Re } g(f))_y = \alpha_r(u, v)u_y + \alpha_s(u, v)v_y \]  

\[ (\text{Im } g(f))_x = \beta_r(u, v)u_x + \beta_s(u, v)v_x \]  

\[ (\text{Im } g(f))_y = \beta_r(u, v)u_y + \beta_s(u, v)v_y. \]  

Substituting the Cauchy-Riemann equations for \( f \) and \( g \),  

\[ u_x = v_y \]  

\[ u_y = -v_x \]  

\[ \alpha_r = \beta_s \]  

\[ \alpha_s = -\beta_r, \]  

it is straightforward to see that \( \text{Re } g(f) \) and \( \text{Im } g(f) \) also satisfy the Cauchy-Riemann equations.
Let $f(z) = z^2$. We’d like to show that $f$ satisfies the Cauchy-Riemann equations. Set $z = x + iy$.

$$(x + iy)^2 = x^2 - y^2 + 2xyi.$$  \hspace{1cm} (12)

$$(\text{Re } f)_x = 2x$$ \hspace{1cm} (13)

$$= (\text{Im } f)_y$$ \hspace{1cm} (14)

$$(\text{Re } f)_y = -2y$$ \hspace{1cm} (15)

$$= -(\text{Im } f)_x.$$ \hspace{1cm} (16)

The $z^3$ case is similar. Just multiply it out and check partials with respect to $x$ and $y$. 

Let $f$ be an analytic function such that $|f|^2 = C$. We’d like to show that $f$ is constant. This is obvious if $C = 0$, so let’s also assume that $C > 0$. Set $f = u + iv$. Then

$$u^2 + v^2 = C. \quad (17)$$

Differentiating, we obtain

$$2uu_x + 2vv_x = 0 \quad (18)$$
$$2uu_y + 2vv_y = 0. \quad (19)$$

$f$ is analytic, so we can use the Cauchy-Riemann equations to see that

$$2uu_x - 2vu_y = 0 \quad (20)$$
$$2uu_y + 2vu_y = 0 \quad (21)$$
$$2uv_y + 2vv_x = 0 \quad (22)$$
$$-2uv_x + 2vv_y = 0. \quad (23)$$

Setting $w_1 = (2u, -2v)$ and $w_2 = (2v, 2u)$, these four equations give

$$\nabla u \cdot w_1 = 0 \quad (24)$$
$$\nabla u \cdot w_2 = 0 \quad (25)$$
$$\nabla v \cdot w_2 = 0 \quad (26)$$
$$\nabla v \cdot w_1 = 0. \quad (27)$$

However, $C > 0$, so at each point $z$ the vectors $w_1$ and $w_2$ are nonzero. They are also orthogonal and therefore a basis of $\mathbb{R}^2$. This allows us to deduce from the previous equations that

$$\nabla u = (0,0) \quad (28)$$
$$\nabla v = (0,0), \quad (29)$$

and we conclude that $f$ is a constant function.