An $n \times n$ matrix $A$ is called a **Markov** matrix if: (1) each entry is $\geq 0$, and (2) the sum of the entries in each column is 1. Markov matrices arise in many applications.

For example, as in the last two homework assignments, you can think of $A$ as describing birds that inhabit $n$ islands: $a_{ij}$ is the fraction of the birds inhabiting island $j$ one year that inhabit island $i$ the following year.

Markov matrices are also used to represent certain random processes. Suppose at each time $t = 0, 1, 2, \ldots$, a system can be in any one of $n$ states. Suppose also that if it is in state $j$ at one time, then the probability of it being in state $i$ at the next time is $p_{ij}$. Then the the probabilities $p_{ij}$ are the entries of a Markov matrix $P$.

In the following problems, suppose that $A$ is an $n \times n$ Markov matrix.

1. Consider a vector $v$ in $\mathbb{R}^n$. Prove that the sum of the entries of $v$ is equal to the sum of the entries of $Av$.

2. Suppose $v$ is an eigenvector of $A$ with eigenvalue $\lambda$. Suppose also that each entry of $v$ is $\geq 0$. Show that $\lambda = 1$.

3. Suppose $w$ is an eigenvector of $A$ with eigenvalue $\mu$. Prove that if $\mu \neq 1$, then $\sum_{i=1}^n w_i = 0$.

Note: problem 2 is relevant in many applications of Markov processes because the vectors of interest will not have any negative entries. (For example, the number of birds on an island cannot be negative.) Also, eigenvectors with eigenvalue 1 are of particular interest because they correspond to equilibria.