SSEA FINAL EXAM – 51 TRACK

• Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
• This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
• In order to receive full credit, please show all of your work and justify your answers. You do not need to simplify your answers unless specifically instructed to do so.
• If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
• Please sign the following:
  “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

Signature:  

The following boxes are strictly for fake grading purposes. Please do not mark.

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(1) (15 points) Answer the following clearly, completely, and concisely.

- Define ‘linear operator.’

- List the qualities a set must have to be a subspace of \( \mathbb{R}^n \).

- For the matrix
  \[
  A = \begin{pmatrix}
    0 & 1 \\
    0 & 0
  \end{pmatrix},
  \]
  the vector \( \overrightarrow{e_1} \) is in both \( N(A) \) and \( C(A) \) (you don’t need to verify this). Motivated by this example, your best friend Jean Claude went on a hunt for other matrices with this property. He claims to have found a \( 2 \times 3 \) matrix \( B \) and vector \( \overrightarrow{v} \) so that \( \overrightarrow{v} \in N(B) \) and \( \overrightarrow{v} \in C(B) \). Explain to Jean Claude why he must be mistaken.
(2) (20 points) Determine whether each statement is true or false; unless otherwise stated, any function below is arbitrary. If the statement is true, cite your reasoning. If it is false, provide a counterexample.

(a) If $A$ is an $r \times c$ matrix, then $N(A) = \{ \overrightarrow{0} \}$ implies $C(A) = \mathbb{R}^r$.

(b) Any subspace of $\mathbb{R}^3$ has infinitely many elements.

(c) If $A$ is an $r \times c$ matrix and $c > r$, then $N(A) \neq \{ \overrightarrow{0} \}$.

(d) The collection $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : xyz = 0 \right\}$ is not a subspace.
(3) (20 points) Let $A$ be the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 2 \\
1 & 1 & -1 & 4
\end{pmatrix}.
$$

- Compute $\text{rref}(A)$.

- Find a basis for $N(A)$ and for $C(A)$. 
• It is a fact that if a vector $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is orthogonal to all vectors in a basis for $\mathcal{C}(A)$, then it is orthogonal to all vectors in $\mathcal{C}(A)$. Use this fact and your basis for $\mathcal{C}(A)$ to write a system of equations which describe conditions on $x_1$, $x_2$ and $x_3$ that are necessary and sufficient to make $\vec{x}$ orthogonal to all vectors in $\mathcal{C}(A)$.
(4) (10 points) Let $A$ be the matrix
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix}.
\]
Find all solutions to the equation $A \vec{x} = \vec{x}$. (Hint: A vector $\vec{x}$ is a solution to $A \vec{x} = \vec{x}$ if and only if $(A - I_3) \vec{x} = \vec{0}$, where $I_3$ is the $3 \times 3$ identity matrix.)
Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator which first rotates vectors through an angle of $\frac{\pi}{2}$ in the counterclockwise direction and then reflects them across the line $x = 0$. Write the matrix for $T$.

Let $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear operator which first reflects vectors across the line $x = 0$ and then rotates them through an angle of $\frac{\pi}{2}$ in the counterclockwise direction. Write the matrix for $S$.

Are $S$ and $T$ the same linear operator? If so, explain carefully. If not, find a vector $\vec{v}$ so that $T(\vec{v}) \neq S(\vec{v})$. 
(6) (10 points) Suppose that $\vec{x}$ and $\vec{y}$ are two vectors so that $\vec{x} - \vec{y}$ and $\vec{x} + \vec{y}$ are orthogonal. Prove that $\|\vec{x}\| = \|\vec{y}\|$. 
(7) (10 points) Suppose that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are nonzero vectors which are mutually orthogonal (i.e., $\mathbf{x}_i \cdot \mathbf{x}_j = 0$ if $i \neq j$). Prove that $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent.