Notes to Sections 2007. 3. 13

§7.3. P520

39. Step 1. Set up the differential equation.

From the problem we have: $\frac{dm}{dt} = km$ and $(mv)' = gm$

$\Rightarrow m'v + mv' = gm \Rightarrow kmv + mv' = gm \Rightarrow kv + v' = g$

$\Rightarrow \frac{dv}{dt} = g - kv$

Step 2. Solve this equation.

This is a separable equation so you should really do it yourself. Solution is: $v = \frac{g}{k} + Ce^{g-kt}$ where C is an arbitrary constant.

Step 3. Take the limit. Since we know $k>0$ from the problem,

$\lim_{t\to\infty} v(t) = \lim_{t\to\infty}\left(\frac{g}{k} + Ce^{g-kt}\right) = \frac{g}{k} + C\cdot 0 = \frac{g}{k}$

41 (a) Step 1. Set up the differential equation.

Problem says the rate is proportional to the product of $\sqrt{A(t)}$ and $M-A(t)$. So for some constants k, $\frac{dA}{dt} = k\sqrt{A}\,(M-A)$

Step 2. We are interested in the maximum of the function $\frac{dA}{dt}$, i.e. we want to find the maximum of the expression $k\sqrt{A}(M-A)$.

The derivative of the function with respect to t is:

$k\sqrt{A}(-1)\frac{dA}{dt} + k(M-A)\frac{1}{2}A^{-\frac{1}{2}}\frac{dA}{dt}$ (chain rule)

$= \frac{1}{2}kA^{-\frac{1}{2}}\frac{dA}{dt}\left[-2A + (M-A)\right]$

$= \frac{1}{2}kA^{-\frac{1}{2}}k\sqrt{A}(M-A)(M-3A)$ (substituting for $\frac{dA}{dt}$ from the equation)

$= \frac{1}{2}k^2(M-A)(M-3A)$

Step 3. Look for the possible maximum.

The above expression is 0 when $M-A=0$ or $M-3A=0$.

But $M-A=0$ never actually occurs, since the area of the tissue is always smaller than the final area M during the growth.

When $M-3A=0$, $A=\frac{M}{3}$. This represents a maximum by the First derivative test, since the expression $\frac{1}{2}k^2(M-A)(M-3A)$ goes from positive to negative when $A(t)=\frac{M}{3}$.

§7.4 P533

21. (a) You can solve this equation by considering it as a separable equation. You can also use a substitution to transform it into an "exponential" equation.

Let $y=P-\frac{m}{k}$, then $\frac{dy}{dt}=\frac{dP}{dt}=kP-M=ky$

$\Rightarrow y=y_0e^{kt} \Rightarrow P-\frac{m}{k}=\left(P_0-\frac{m}{k}\right)e^{kt} \Rightarrow P(t)=\frac{m}{k}+\left(P_0-\frac{m}{k}\right)e^{kt}$

(b) Since $k>0$, there will be an exponential expasion

$\Leftrightarrow P_0-\frac{m}{k}>0 \Leftrightarrow m<kP_0$

(c) The population will be constant if $P_0-\frac{m}{k}=0 \Leftrightarrow m=kP_0$

It will decline if $P_0-\frac{m}{k}<0 \Leftrightarrow m>kP_0$

(d) $P_0=8000000$, $k=\alpha-\beta=0.016$, $m=210000$

Compare m $(=210000)$ and kP_0 $(=128000)$, we know $m>kP_0$, So by part (c), the population was declining.

§7.5 P542

1. (a) $\frac{dP}{dt}=0.05P-0.0005P^2=0.05P(1-0.01P)=0.05\cdot P\cdot\left(1-\frac{P}{100}\right)$

So we see that carrying capacity is $K=100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P=50$. The solutions are increasing for $0<P_0<100$ and decreasing for $P_0>100$.

(c) (I won't draw the solutions here.)

All of the six solutions approach $P=100$ as t increases. As in part (b), the solution differ since for $0<P_0<100$ they are increasing, and for $P_0>100$ they are decreasing. Also, some have inflection points and some don't. It appears that the solutions which have $P_0=20$ and $P_0=40$ have inflection points at $P=50$.

(d) The equilibrium solutions are $P=0$ and $P=100$. The increasing solutions move away from $P=0$ and all nonzero solutions approach $P=100$ as $t\to\infty$.

11 (a) The term -15 represents a harvesting of fish at a constant rate, in this case, 15 fish per week. This is the rate at which fish are caught.

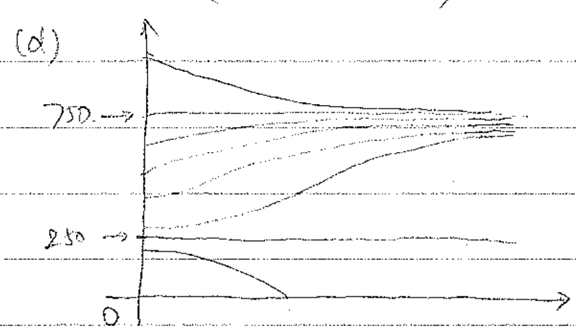
(b) Do it yourself!

(c) From the graph in part (b), it appears that $P(t)=250$ and $P(t)=750$ are the equilibrium solutions. We confirm this analytically by solving the equation $\frac{dP}{dt}=0$ as follows: $0.08P(1-\frac{P}{1000})-15=0$

$\Rightarrow 0.08P-0.00008P^2-15=0 \Rightarrow -0.00008(P^2-1000P+187500)=0$

$\Rightarrow (P-250)(P-750)=0 \Rightarrow P=250$ or 750

(d)



For $0<P_0<250$, $P(t)$ decreases to 0.
For $P_0=250$, $P(t)$ remains constant.
For $250<P_0<750$, $P(t)$ increases and approaches 750.
For $P_0=750$, $P(t)$ remains constants.
For $P_0>750$, $P(t)$ decreases and approaches 750.

(e) You can use either the partial fractions or the substitution.
Here's how you use the substitution:
Factor RHS as the product of two factors:

$$0.08P(1-\frac{P}{1000})-15=-0.00008(P^2-1000P+187500)$$

$$=-0.00008(P-250)(P-750)=0.00008(P-250)(750-P)$$

Rewrite the differential equation as:

$$\frac{dP}{dt} = 0.00008\,(P-250)(750-P)$$

Substitution: $y = P - 250$, then

$$\frac{dy}{dt} = \frac{dP}{dt} = 0.00008\,(P-250)(750-P)$$
$$= 0.00008 \cdot y \cdot [750 - (y+250)]$$
$$= 0.00008 \cdot y \cdot (500-y) = 0.04 \cdot y \cdot \left(1 - \frac{y}{500}\right)$$

Quote the solution formula for logistic equations:

$$y = \frac{K}{1 + Ae^{-kt}} \quad \text{where } A = \frac{K - y_0}{y_0}$$

Note that $k = 0.04$, $K = 500$, $P_0 = 200$ and 300 respectively

when $P_0 = 200$, $y_0 = -50$, $A = \frac{550}{-50} = -11$

so $y(t) = \frac{500}{1 - 11e^{-0.04t}}$ $\Rightarrow$ $P(t) = y(t) + 250 = \frac{500}{1-11e^{-0.04t}} + 250$

when $P_0 = 300$, $y_0 = 50$, $A = \frac{450}{50} = 9$

so $y(t) = \frac{500}{1+9e^{-0.04t}}$ $\Rightarrow$ $P(t) = y(t) + 250 = \frac{500}{1+9e^{-0.04t}} + 250$

13. (a) $\frac{dP}{dt} = kP \cdot \left(1 - \frac{P}{K}\right) \cdot \left(1 - \frac{m}{P}\right)$

$+ \quad + \quad +$ when $m < P < K$ $\Rightarrow$ $\frac{dP}{dt} > 0$

$+ \quad + \quad -$ when $0 < P < m$ $\Rightarrow$ $\frac{dP}{dt} < 0$

So when $m < P < K$, P is increasing

when $0 < P < m$, P is decreasing

(b) (I won't draw graphs here.)

$k = 0.08$, $K = 1000$, $m = 200$ $\Rightarrow$ $\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$

For $0 < P_0 < 200$, the population dies out.

For $P_0 = 200$, the population is steady.

For $200 < P_0 < 1000$, the population increases and approaches 1000.

For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t)=200$ and $P(t)=1000$.

(c) Use substitution.

Rewrite the differential equation as

$$\frac{dP}{dt}=k(P-m)\left(1-\frac{P}{K}\right)=\frac{k}{K}(P-m)(K-P)$$

Set $y=P-m$, then $\frac{dy}{dt}=\frac{dP}{dt}=\frac{k}{K}(P-m)(K-P)$

$$=\frac{k}{K}\cdot y\cdot[K-(y+m)]=\frac{k}{K}\cdot y\cdot(K-m-y)=\frac{k}{K}(K-m)\cdot y\cdot\left(1-\frac{y}{K-m}\right)$$

Apply the solution formula for logistic equations:

$$y(t)=\frac{K-m}{1+Ae^{-\frac{k}{K}(K-m)t}} \quad \text{where } A=\frac{(K-m)-(P_0-m)}{P_0-m}=\frac{K-P_0}{P_0-m}$$

$$\text{So } P(t)=y(t)+m=\frac{K-m}{1+\frac{K-P_0}{P_0-m}e^{-\frac{k}{K}(K-m)t}}+m=\frac{m(K-P_0)+K(P_0-m)e^{\frac{k}{K}(K-m)t}}{K-P_0+(P_0-m)e^{\frac{k}{K}(K-m)t}}$$

(d) If $P_0<m$, then $P_0-m<0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0)=P_0(K-m)>0$ and

$P_0-m<0 \iff \lim_{t\to\infty} K(P_0-m)e^{\frac{k}{K}(K-m)t}=-\infty \Rightarrow \lim_{t\to\infty} N(t)=-\infty$

Since N is continuous, there is a number t such that $N(t)=0$ and thus $P(t)=0$. So the species will become extinct at some point.

9. (a) $\frac{dP}{dt}=kP\left(1-\frac{P}{K}\right) \Rightarrow \frac{d^2P}{dt^2}=k\left[P\left(-\frac{1}{K}\right)\frac{dP}{dt}+\left(1-\frac{P}{K}\right)\frac{dP}{dt}\right]$

$$=k\frac{dP}{dt}\left(-\frac{P}{K}+1-\frac{P}{K}\right)=k\cdot\left[kP\left(1-\frac{P}{K}\right)\right]\left(1-\frac{2P}{K}\right)=k^2P\left(1-\frac{P}{K}\right)\left(1-\frac{2P}{K}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P''=0$.

From part (a), $P''=0 \iff P=0$ or $P=K$ or $P=K/2$

Since $0<P<K$, we see that $P''=0 \iff P=K/2$.

Then use either the first derivative test or the second derivative test to deduce that when $P=\frac{K}{2}$, P' reaches its maximum.