Notes for Sections 2007.3.8

§ 7.3 P519 (I will do 7, 9, 33 in sections.)

3. $(x^2+1)y' = xy \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2+1} \Rightarrow \frac{dy}{y} = \frac{x\,dx}{x^2+1}$ (when $y \neq 0$)

$\Rightarrow \int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1} \Rightarrow \ln|y| = \frac{1}{2}\ln(x^2+1) + C = \ln(x^2+1)^{1/2} + \ln e^C$

$= \ln(e^C\sqrt{x^2+1})$

$\Rightarrow |y| = e^C\sqrt{x^2+1} \Rightarrow y = K\sqrt{x^2+1}$ where $K = \pm e^C$ is a constant

(In our derivation, K was nonzero, but we can restore the excluded case $y=0$ by allowing K to be zero.)

7. $\frac{du}{dt} = 2+2u+t+tu \Rightarrow \frac{du}{dt} = (1+u)(2+t) \Rightarrow \int \frac{du}{1+u} = \int (2+t)\,dt \quad (u \neq -1)$

$\Rightarrow \ln|1+u| = 2t + \frac{1}{2}t^2 + C \Rightarrow |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}$ where $K = e^C$

$\Rightarrow 1+u = \pm Ke^{t^2/2+2t} \Rightarrow u = -1 \pm Ke^{t^2/2+2t}$ where $K>0$

$u=-1$ is also a solution, so $u = -1 + Ae^{t^2/2+2t}$ where A is an arbitrary constant.

9. $\frac{du}{dt} = \frac{2t+\sec^2 t}{2u}$, $u(0) = -5$

$2u\,du = (2t+\sec^2 t)\,dt \Rightarrow u^2 = t^2 + \tan t + C$

Plug in $t=0$ and $u=-5$, we have $(-5)^2 = 0^2 + \tan 0 + C \Rightarrow C = 25$

Therefore $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2+\tan t+25}$

Since $u(0) = -5$, we have $u = -\sqrt{t^2+\tan t+25}$

29. $\frac{dP}{dt} = k(M-P) \Leftrightarrow \int \frac{dP}{M-P} = \int k\,dt \Leftrightarrow -\ln|P-M| = kt + C$

$\Leftrightarrow |P-M| = e^{-kt-C} \Leftrightarrow P-M = Ae^{-kt}$ where $A = \pm e^{-C}$

$\Leftrightarrow P = M + Ae^{-kt}$ where $A \neq 0$ is a constant.

We can still restore the lost solution $P=M$ by allowing A to be any constant including 0.

$\lim_{t\to\infty} P(t) = M + A\cdot 0 = M$

33. (a) $\frac{dc}{dt} = r - kC \Rightarrow \frac{dc}{dt} = -(kc - r) \Rightarrow \int \frac{dc}{kc-r} = \int -dt$

$\Rightarrow \frac{1}{k}\ln|kC - r| = -t + M_1 \Rightarrow \ln|kc - r| = -kt + M_2$

$\Rightarrow |kc - r| = e^{-kt+M_2} \Rightarrow kc - r = M_3 e^{-kt} \quad (M_3 \neq 0)$

$\Rightarrow kc = M_3 e^{-kt} + r \Rightarrow C(t) = M_4 e^{-kt} + \frac{r}{k}$ (where $M_4 \neq 0$)

By restoring the lost solution we allow M_4 to be chosen arbitrarily.
The initial condition gives $C(0) = C_0 \Rightarrow C_0 = M_4 e^{-k\cdot 0} + \frac{r}{k}$

$\Rightarrow M_4 = C_0 - \frac{r}{k} \Rightarrow C(t) = (C_0 - \frac{r}{k})e^{-kt} + \frac{r}{k}$

(b) If $C_0 < \frac{r}{k}$, then $C_0 - \frac{r}{k} < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t) = \frac{r}{k}$. As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of $\frac{r}{k}$ increases.

§7.4 P532 (I will do 3, 11 in sections.)

In order to set up a differential equation, look for the most important sentence in the problem which relates two quatities in a certain way.

3. Idea: most important sentence: "grows at a rate proportional to its size"
translation: "rate of growth = constant × population size"
i.e. "derivative of population with respect to time = constant × population"

differential equation: $\frac{dP}{dt} = kP$

(a) By Theorem 2, $P(t) = P(0)e^{kt} = 100e^{kt}$

Now $P(1) = 100e^{k} = 420 \Rightarrow e^k = \frac{420}{100} \Rightarrow k = \ln 4.2$

So $P(t) = 100e^{\ln 4.2 \cdot t} = 100 \cdot 4.2^t$

(b) $P(3) = 100 \cdot 4.2^3 = 7408.8 \approx 7409$ bacteria

(c) $P' = kP \Rightarrow P'(3) = kP(3) = \ln 4.2 \cdot 100 \cdot 4.2^3 \approx 10632$ bacteria/hour

(d) $P(t) = 100 \cdot 4.2^t = 10000 \Rightarrow 4.2^t = 100 \Rightarrow t = \ln 100 / \ln 4.2 \approx 3.2$ hours.

5. (a) We measure the time t in years and let $t=0$ in the year 1750. We measure the population $P(t)$ in millions of people.
The initial condition is $P(0)=790$.
By using the exponential growth model, $P(t)=P(0)e^{kt}$.
We can find the value for k by using the fact that population in 1800 ($t=50$) was 980. $P(50)=790\cdot e^{k\cdot 50}=980$
$\Rightarrow e^{50k}=\frac{980}{790} \Rightarrow 50k=\ln\frac{980}{790} \Rightarrow k=\frac{1}{50}\ln\frac{980}{790}\approx 0.0043104$.
So with the model, we have:
in the year 1900, $t=1900-1750=150$ $P(150)=790\cdot e^{k\cdot 150}\approx 1508$ million
in the year 1950, $t=1950-1750=200$, $P(200)=790\cdot e^{k\cdot 200}\approx 1871$ million
Both of these estimates are too low.

(b) Similarly, in this part we let $t=0$ in the initial year 1850.
$P(t)=P(0)\cdot e^{kt}=1260\cdot e^{kt}$
Since $P(50)=1650$, we have $1650=1260\cdot e^{k\cdot 50}$
$\Rightarrow k=\frac{1}{50}\ln\frac{1650}{1260}\approx 0.005393$
With this model, we estimate in the year 1950,
$P(100)=1260\cdot e^{k\cdot 100}\approx 2161$ million
This is still too low, but closer than the estimate in part (a).

(c) In this part we let $t=0$ in the initial year 1900
$P(t)=P(0)\cdot e^{kt}=1650\cdot e^{kt}$
Since $P(50)=2560$, we have $2560=1650\cdot e^{k\cdot 50}$
$\Rightarrow k=\frac{1}{50}\ln\frac{2560}{1650}\approx 0.008785$
With this model, we estimate in the year 2000,
$P(100)=1650\cdot e^{k\cdot 100}\approx 3972$ million
This is much too low.

The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constants.

11. Idea: most important sentence in the problem: "the level of radioactivity must also decay exponentially", "with a half-life of about 5730 years".

half-life = time required for the level of radioactivity to reduce to half of its initial amount

Let $y(t)$ be the level of radioactivity. Thus

$y(t) = y(0)e^{kt}$ and t is determined by using the half-life:

$y(5730) = \frac{1}{2}y(0) \Rightarrow \frac{1}{2}y(0) = y(0)e^{k\cdot 5730} \Rightarrow e^{5730k} = \frac{1}{2}$

$\Rightarrow 5730k = \ln\frac{1}{2} \Rightarrow k = \frac{1}{5730}\ln\frac{1}{2} = -\frac{\ln 2}{5730}$

If 74% of the ^{14}C remains, then we know that $y(t) = 0.74\,y(0)$

$\Rightarrow 0.74\,y(0) = y(0)e^{-\frac{\ln 2}{5730}t} \Rightarrow -\frac{\ln 2}{5730}t = \ln 0.74$

$\Rightarrow t = -\frac{5730\ln 0.74}{\ln 2} \approx 2489$ years.

17. Let $p(h)$ be the pressure at altitude h. Then $\frac{dp}{dt} = kp$

$\Rightarrow p(h) = p(0)e^{kh} = 101.3\cdot e^{kh}$

$p(1000) = 101.3\,e^{k\cdot 1000} = 87.14 \Rightarrow 1000k = \ln\frac{87.14}{101.3} \Rightarrow k = \frac{1}{1000}\ln\frac{87.14}{101.3}$

$\Rightarrow p(h) = 101.3\cdot e^{\frac{1}{1000}\ln\frac{87.14}{101.3}\cdot h}$

(a) $p(3000) = 101.3\cdot e^{\frac{1}{1000}\ln\frac{87.14}{101.3}\cdot 3000} = 101.3\cdot e^{3\ln\frac{87.14}{101.3}} \approx 64.5$ kPa

(b) $p(6187) = 101.3\,e^{\frac{1}{1000}\ln\frac{87.14}{101.3}\cdot 6187} \approx 39.9$ kPa