Notes to Sections 2007. 2. 27.

§8.7. P615

3. Equation 7 gives the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} (n+1) x^n$$

Applying the ratio test gives the radius of convergence $R=1$

(Fill out the details yourself.)

5. Similar to Example 4 in the text.

23. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \Rightarrow f(x) = x^2 e^{-x} = x^2 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{n+2} \qquad R=\infty$$

31. $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \text{ so } e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!} 0.2^2 - \frac{1}{3!} 0.2^3 + \frac{1}{4!} 0.2^4$$

$$- \frac{1}{5!} 0.2^5 + \frac{1}{6} 0.2^6 - \cdots$$

Notice that $\frac{1}{6!} 0.2^6 = 8.\bar{8} \times 10^{-8}$ so by the Alternating series estimation theorem, $e^{-0.2} \approx \sum_{n=0}^{5} \frac{(-0.2)^n}{n!} \approx 0.81873$, correct to 5 decimal places.

41 $$\lim_{x\to 0} \frac{x - \tan^{-1} x}{x^3} = \lim_{x \to 0} \frac{x - (x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{1}{3} - \frac{1}{5}x^2 + \frac{1}{7}x^4 - \cdots\right) = \frac{1}{3}$$

43. $$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots) - x + \frac{1}{6}x^3}{x^5}$$

$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{1}{7!}x^2 + \cdots\right) = \frac{1}{5!} = \frac{1}{120}$$

49. $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$$

51. $$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{\pi}{4})^{2n+1}}{(2n+1)!} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

§ 8.9 P628

11. (a) $f(x) = \sqrt{x} \approx T_2(x)$	n	$f^{(n)}(x)$	$f^{(n)}(4)$
$= 2 + \frac{1}{4}(x-4) - \frac{1}{2!}\frac{1}{32}(x-4)^2$	0	$\sqrt{x}$	2
$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$	1	$\frac{1}{2}x^{-\frac{1}{2}}$	$\frac{1}{4}$
(b) $\lvert R_2(x)\rvert \le \frac{M}{3!}\lvert x-4\rvert^3$	2	$-\frac{1}{4}x^{-\frac{3}{2}}$	$-\frac{1}{32}$
where $\lvert f^{(3)}(x)\rvert \le M$	3	$\frac{3}{8}x^{-\frac{5}{2}}$	

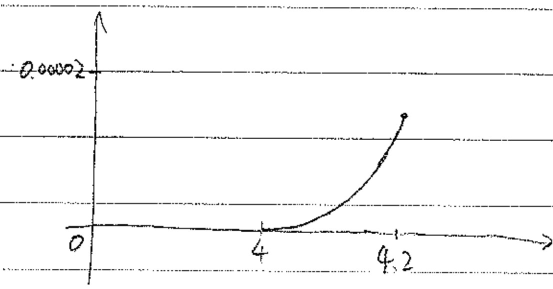
Now $4 \le x \le 4.2 \Rightarrow |x-4| \le 0.2 \Rightarrow |x-4|^3 \le 0.008$.

Since $f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}}$ is decreasing on $[4, 4.2]$, we can take

$M = |f^{(3)}(4)| = \frac{3}{8} \cdot 4^{-\frac{5}{2}} = \frac{3}{256}$,

So $|R_2(x)| \le \frac{1}{3!} \cdot \frac{3}{256} \cdot 0.008 = 0.000015625$

(c) From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$ on the right, it seems that the error is less than 1.52×10^{-5} on $[4, 4.2]$



21. All derivatives of e^x are e^x, so $|R_n(x)| \le \frac{M}{(n+1)!}|x|^{n+1}$ where $0 \le x \le 0.1$ and $|f^{(n+1)}(x)| = |e^x| \le M$ on $[0, 0.1]$

So take $M = e^{0.1}$ we have $|R_n(x)| \le \frac{e^{0.1}}{(n+1)!}|x|^{n+1}$.

Letting $x = 0.1$, $|R_n(0.1)| \le \frac{e^{0.1}}{(n+1)!} 0.1^{n+1} < 0.00001$, and by trial and error we find that $n=3$ satisfies this inequality since $\frac{e^{0.1}}{(3+1)!} 0.1^{3+1} \approx 0.0000046 < 0.00001$

Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2, 3$, we can estimate $e^{0.1}$ to within 0.00001.

23. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$

By the alternating series estimation theorem, the error in the approximation $\sin x \approx x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \Leftrightarrow |x^5| < 120 \cdot 0.01 \Leftrightarrow |x| < (1.2)^{\frac{1}{5}} \approx 1.037$

Thus the desired range of values for x is $-1.037 < x < 1.037$.

(I would like to skip the graph part. Ask me if you want to see the graph.)

25. Let $s(t)$ be the position function of the car, and for convenience set $s(0)=0$. The velocity of the car is $v(t)=s'(t)$ and the acceleration is $a(t)=s''(t)$, so the second degree Taylor polynomial is $T_2(t)=s(0)+v(0)t+\frac{a(0)}{2!}t^2 = 20t+t^2$.
We estimate the distance travelled during the next second to be $s(1)\approx T_2(1) = 20+1=21$ m
The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an accelaration of $2m/s$ for that long (if it did, its final speed would be 140 m/s ≈ 313 mi/h which is impossible.)