Notes to Sections 2007.2.22

§8.6. P604

3. Our goal is to write the function in the form $\frac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series.

$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$ with $|-x|<1 \iff |x|<1$

so $R=1$ and $I=(-1,1)$

9 $f(x) = \frac{x}{9+x^2} = \frac{x}{9} \frac{1}{1+\frac{x^2}{9}} = \frac{x}{9} \frac{1}{1-(-\frac{x^2}{9})} = \frac{x}{9} \cdot \sum_{n=0}^{\infty} \left(-\frac{x^2}{9}\right)^n$

$= \frac{x}{9} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{9^n} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}}$

The geometric series $\sum_{n=0}^{\infty} \left(-\frac{x^2}{9}\right)^n$ converges when $\left|-\frac{x^2}{9}\right|<1$

$\iff \left|\frac{x^2}{9}\right|<1 \iff \frac{x^2}{9}<1 \iff x^2<9 \iff -3<x<3$

so $R=3$ and $I=(-3,3)$

11 (a) $f(x) = \frac{1}{(1+x)^2} = -\frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n x^n$ (from exercise 3)

$= -\underbrace{\sum_{n=0}^{\infty} \frac{d}{dx}\left[(-1)^n x^n\right]}_{\text{(term by term differentiation)}} = -\sum_{n=0}^{\infty} (-1)^n n x^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1}$

To write the power series with x^n rather than x^{n-1}, we can decrease the initial value of the summation variable n by 1 and increase each occurence of n in the term by 1. So the expansion can also be written as

$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} n x^{n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} = \sum_{n=0}^{\infty} (-1)^{n+2} (n+1) x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$

(n=0 is useless because it gives a term which is 0)

(decrease the initial value and increase each occurence of n by 1)

(notice that $(-1)^{n+2} = (-1)^n \cdot (-1)^2 = (-1)^n \cdot 1 = (-1)^n$)

By theorem 2, the radius of convergence is the same as that of the expansion of $\frac{1}{1+x}$ which is $R=1$

(b) $f(x) = \frac{1}{(1+x)^3} = -\frac{1}{2}\frac{d}{dx}\left(\frac{1}{(1+x)^2}\right) = -\frac{1}{2}\frac{d}{dx}\left(\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right)$

$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{d}{dx}\left((-1)^n(n+1)x^n\right) = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+1)n\,x^{n-1}$

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)n\,x^{n-1} = -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}(n+2)(n+1)x^n$

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+2}(n+2)(n+1)x^n = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R=1$.

(c) $f(x) = \frac{x^2}{(1+x)^3} = x^2\cdot\frac{1}{(1+x)^3} = x^2\cdot\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$

$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$

Similarly, to write the power series with x^n rather than x^{n+2}, we will decrease each occurence of n in the term by 2 and increase the initial value of the summation variable by 2. This gives us

$f(x) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2} = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^{n-2}n(n-1)x^n = \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1)x^n$

13. $f(x) = \ln(5-x) = -\int\frac{1}{5-x}dx = -\frac{1}{5}\int\frac{1}{1-\frac{x}{5}}dx = -\frac{1}{5}\int\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n dx$

$= -\frac{1}{5}\sum_{n=0}^{\infty}\int\left(\frac{x}{5}\right)^n dx = -\frac{1}{5}\sum_{n=0}^{\infty}\int\frac{x^n}{5^n}dx = -\frac{1}{5}\sum_{n=0}^{\infty}\frac{1}{5^n}\frac{x^{n+1}}{n+1}+C$

$= -\sum_{n=0}^{\infty}\frac{1}{5^{n+1}}\frac{x^{n+1}}{n+1}+C = -\sum_{n=1}^{\infty}\frac{1}{5^n}\frac{x^n}{n}+C$

putting $x=0$ we get $C=\ln 5$, so

$f(x) = -\sum_{n=1}^{\infty}\frac{1}{5^n}\frac{x^n}{n}+\ln 5$

The radius of convergence is the same as the radius of convergence when expanding $\frac{1}{1-\frac{x}{5}}$, so $\left|\frac{x}{5}\right|<1 \iff |x|<5$ i.e. $R=5$

29. From Example 6, $\ln(1-x) = -\sum_{n=0}^{\infty}\frac{x^n}{n}$ with $x=-0.1$ in our case.

So we have $\ln 1.1 = \ln(1-(-0.1)) = -\sum_{n=0}^{\infty}\frac{(-0.1)^n}{n} = \sum_{n=0}^{\infty}\frac{(-0.1)^{n+1}}{n}$

which is an alternating series which satisfies the two conditions in the alternating series test. So the formula for the error bound is valid. Now we have

$\ln 1.1 = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$

If we only take the partial sum of the first 3 terms, the error is at most $\frac{0.0001}{4} = 0.000025 > 10^{-5}$ (i.e the error is not controlled as we want.)

If we take the partial sum of the first 4 terms, the error is at most $\frac{0.00001}{5} = 0.000002 < 10^{-5}$

So $\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$

33 (a) By ratio test, it's not hard to find that the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is ∞. (You should verify this yourself!) So the term-by-term differentiation is valid for any x.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow f'(x) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \frac{d}{dx}\left(\frac{x^0}{0!}\right) + \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right)$$

$$= 0 + \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so $f'(x) = f(x)$ because they have the same power series expansion!

35. Consider $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. by the ratio test, $a_n = \frac{x^n}{n^2}$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \frac{n^2}{x^n}\right| = \lim_{n\to\infty} \left|x \cdot \left(\frac{n}{n+1}\right)^2\right| = |x| < 1 \text{ for convergence}$$

So the radius of convergence $R = 1$.

For the endpoints, when $x=1$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series. When $x=-1$, the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is convergent by the alternating series test. So the interval of convergence is $[-1, 1]$

By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n^2} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = \sum_{n=0}^{\infty} \frac{x^n}{n+1}$ and it diverges for $x=1$ (harmonic series) and converges for $x=-1$ (alternating series test), so the interval is $[-1, 1)$. $f''(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n+1} = \sum_{n=0}^{\infty} \frac{(n+1)x^n}{n+2}$ diverges at both 1 and -1 (Test for Divergence). So the interval is $(-1, 1)$