Notes for Sections 2007.2.20

§ 8.5 P598.

5. $a_n = \frac{(-1)^{n-1} x^n}{n^3}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-1)^{n-1} x^n} \right| = \lim_{n\to\infty} \left| \frac{(-1) x n^3}{(n+1)^3} \right|$

$= \lim_{n\to\infty} |x| \cdot \left(\frac{n}{n+1}\right)^3 = |x|$

By ratio test, the radius of convergence $R=1$.

Check the endpoints.

$x=1$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test.

$x=-1$ $\sum_{n=1}^{\infty} (-1)\cdot\frac{1}{n^3}$ converges because it is a constant multiple of a convergent p-series $(p=3>1)$

So the interval of convergence $I = [-1, 1]$

7 $a_n = \frac{x^n}{n!}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left|\frac{x}{n+1}\right| = 0$ for all x

By ratio test, the radius of convergence $R=\infty$

the interval of convergence $I = (-\infty, +\infty)$

9. $a_n = \frac{(-2)^n x^n}{\sqrt[4]{n}}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-2)^{n+1} x^{n+1}}{\sqrt[4]{n+1}} \cdot \frac{\sqrt[4]{n}}{(-2)^n x^n} \right| = \lim_{n\to\infty} \left| \frac{(-2) x \sqrt[4]{n}}{\sqrt[4]{n+1}} \right|$

$= \lim_{n\to\infty} 2|x| \cdot \sqrt[4]{\frac{n}{n+1}} = 2|x|$

By ratio test, the series converges when $2|x|<1 \Leftrightarrow |x|<\frac{1}{2}$, so $R=\frac{1}{2}$.

Check the endpoints:

$x=\frac{1}{2}$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$ Converges by the Alternating Series Test.

$x=-\frac{1}{2}$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ diverges because it's a p-series with $p=\frac{1}{4}\leq 1$.

So the interval of convergence $I = (-\frac{1}{2}, \frac{1}{2}]$.

13. $a_n = (-1)^n \frac{(x+2)^n}{n\cdot 2^n}$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x+2)^{n+1}}{(n+1)\cdot 2^{n+1}} \cdot \frac{n\cdot 2^n}{(-1)^n\cdot(x+2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(-1)(x+2)\cdot n}{(n+1)\cdot 2} \right| = \frac{1}{2}|x+2|$

By the ratio test, the series converges when $\frac{|x+2|}{2} < 1 \Leftrightarrow |x+2| < 2$

$\Leftrightarrow -2 < x+2 < 2 \Leftrightarrow -4 < x < 0$ So $R=2$

Check the endpoints:

$x=0$ $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test.

$x=-4$ $\sum_{n=1}^{\infty} (-1)^n \frac{(-2)^n}{n \cdot 2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series.

So interval of convergence $I=(-4, 0]$

17. $a_n = n!(2x-1)^n$ then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty} |(n+1)(2x-1)|$

$\to \infty$ as $n\to\infty$ for all $2x-1 \neq 0 \iff x \neq \frac{1}{2}$

Since the series diverges for all $x \neq \frac{1}{2}$, radius of convergence $R = \frac{1}{2}$ and interval of convergence $= \{\frac{1}{2}\}$.

19. (a) We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x=4$. So by theorem 3, the radius of convergence is at least 4 and the series must converge for at least $-4 < x \leq 4$. In particular, it converges when $x=-2$, that is $\sum_{n=0}^{\infty} c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty} c_n(-4)^n$ is necessarily convergent. (See the comments after Theorem 3 about convergence at the endpoints of an interval. An example is $c_n = \frac{(-1)^n}{n \cdot 4^n}$.)