Notes to Sections 2007.2.15.

Here are all tests we have until now:

(1) Divergence test ($\lim a_n \neq 0$)

(2) Integral test

(3) Comparison test

(4) Limit comparison test

$\Big\}$ for positive series

(5) Alternating Series convergence test

(6) Ratio test for absolute convergence.

How do you know which test to use when given a series? Please read the file I mentioned. (The link is to the right of the link of this note.)

§8.4 P592

5. Alternating series. $b_n = \frac{1}{\sqrt{n}}$ decreasing and $\lim_{n\to\infty} b_n = 0$
So the series is convergent.

13. Alternating series. $b_n = \frac{1}{n^6}$ decreasing and $\lim_{n\to\infty} b_n = 0$
So the series is convergent.
By the error estimate formula, notice that $b_5 = \frac{1}{5^6} = 0.000064 > 0.00005$ and $b_6 = \frac{1}{6^6} \approx 0.00002 < 0.00005$, so n is at least 5. (That is, since the 6th term is less than the desired error, we need to add the first 5 terms to get the sum to the desired accuracy.)

21. Ratio test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-10)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-10)^n}\right| = \lim_{n\to\infty}\left|\frac{-10}{n+1}\right| = 0 < 1$
So the series is absolutely convergent.

11. If $p>0$, $\frac{1}{n^p}$ is decreasing and $\lim_{n\to\infty}\frac{1}{n^p} = 0$, so convergent by alternating series test.
If $p \le 0$ $\lim_{n\to\infty}\frac{1}{n^p}$ does not exist. So divergent by test for divergence in §8.2.

25. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{10^{n+1}}{(n+2)\cdot 4^{2n+3}}\cdot\frac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim_{n\to\infty}\frac{10}{4^2}\frac{n+1}{n+2} = \frac{5}{8} < 1$

so the series is absolutely convergent.

(Rmk: Since all terms of the series are positive, absolute convergence is the same as convergence.)

29. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n\,1\cdot3\cdot5\cdot\cdots\cdot(2n-1)(2n+1)}{(2n+1)!}\cdot\frac{(2n-1)!}{(-1)^{n-1}\,1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}\right|$

$= \lim_{n\to\infty}\left|\frac{(-1)^n}{(-1)^{n-1}}\cdot\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)(2n+1)}{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}\cdot\frac{(2n-1)!}{(2n+1)!}\right| = \lim_{n\to\infty}\left|(-1)\cdot(2n+1)\cdot\frac{1}{(2n+1)\cdot 2n}\right|$

$= \lim_{n\to\infty}\frac{1}{2n} = 0 < 1$ so the series is absolutely convergent.

33. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1/(n+1)^3}{1/n^3}\right| = \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = \lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^3} = 1$. Inconclusive!

(b) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right| = \lim_{n\to\infty}\frac{n+1}{2n} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}$ Convergent!

(c) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-3)^{n-1}}\right| = \lim_{n\to\infty}\left|\frac{(-3)\sqrt{n}}{\sqrt{n+1}}\right| = 3\lim_{n\to\infty}\sqrt{\frac{n}{n+1}}$

$= 3\lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}} = 3\cdot\sqrt{\frac{1}{1+0}} = 3$ Divergent!

(d) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}}{1+(n+1)^2}\cdot\frac{1+n^2}{\sqrt{n}}\right| = \lim_{n\to\infty}\left(\frac{\sqrt{n+1}}{\sqrt{n}}\cdot\frac{1+n^2}{1+(n+1)^2}\right)$

$= \lim_{n\to\infty}\left(\sqrt{1+\frac{1}{n}}\cdot\frac{\frac{1}{n^2}+1}{\frac{1}{n^2}+(1+\frac{1}{n})^2}\right) = \sqrt{1+0}\cdot\frac{0+1}{0+(1+0)^2} = 1$ Inconclusive!

Three more solutions to daily homework problems in § 8.3.
(upon request.) (on page 585)

17. $\frac{\cos^2 n}{n^2+1} \le \frac{1}{n^2+1} < \frac{1}{n^2}$ Since the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent $(p=2>1)$,
(comparison test) the new series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{1+n^2}$ is also convergent.

19. $\frac{n-1}{n \cdot 4^n}$ is positive for $n>1$ and $\frac{n-1}{n \cdot 4^n} < \frac{n}{n \cdot 4^n} = \frac{1}{4^n} = \left(\frac{1}{4}\right)^n$ (comparison test)

Since the geometric series $\sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n$ is convergent $(r=\frac{1}{4})$,
the new series $\sum_{n=1}^{\infty} \frac{n-1}{n \cdot 4^n}$ is also convergent.

21. (Limit comparison) $a_n = \frac{1}{\sqrt{n^2+1}}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = \frac{1}{\sqrt{1+0}} = 1$$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, so is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.