Notes to Sections 2007. 2. 13

A few things you need to remember in §8.2 and §8.3.

1. Important series and their convergence / divergence.

(1) Geometric series: $\sum_{n=1}^{\infty} ar^{n-1}$ (or written as $\sum_{n=0}^{\infty} ar^n$)

divergent when $|r| \geq 1$; convergent when $|r| < 1$, sum $= \frac{a}{1-r}$

(2) Harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

(Note that it is a special case of p-series.)

(3) p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$

divergent when $p \leq 1$; convergent when $p > 1$

2. Tests for convergence / divergence

(1) If $\lim_{n\to\infty} a_n$ doesn't exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(2) If $a_n = f(n)$ where f is continuous, positive, ultimately decreasing on $[1, +\infty)$, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{+\infty} f(x)dx$ have the same convergence / divergence.

(3) (Comparison I). $\sum a_n$, $\sum b_n$ are series with positive terms, $a_n \leq b_n$ for all n.

then: If $\sum b_n$ is convergent, then $\sum a_n$ is convergent;

If $\sum a_n$ is divergent, then $\sum b_n$ is divergent.

(4) (Comparison II). $\sum a_n$, $\sum b_n$ are series with positive terms.

If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ where c is positive and finite,

then $\sum a_n$, $\sum b_n$ have the same convergence / divergence

3. Miscellaneous

(1) Remainder estimates for integral test

$$s_n + \int_{n+1}^{\infty} f(x)dx \leq s \leq s_n + \int_n^{\infty} f(x)dx$$

(2) Properties of convergent series (on page 573, formulae 8)

(3) A finite number of terms doesn't affect the convergence / divergence.

(4) In practice, the new series we use in comparison tests are those listed above, namely geometric series, harmonic series or p-series.

(5) A special technique : see Example 6 on page 571

The general case is: if the denominator of a typical term can be factored as the product of two factors whose difference is a constant number, this technique works. For example :

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2} \quad \text{or} \quad \frac{2}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}$$

$$\frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3} \quad \text{or} \quad \frac{3}{(n-1)(n+2)} = \frac{1}{n-1} - \frac{1}{n+2}$$

However, there are still some exceptional cases in which you can use this technique, (for example, exercise 30 on page 574).

§ 8.2 P574

$\boxed{17, 21, 23, 27}$ I will talk about these but they are not hard.

49 The series $1-1+1-1+1-1+\cdots$ divergers (geometric series with $r=-1$) so we cannot say it's equal to 0 or 1.

§ 8.3. P585

$\boxed{3, 5, 7, 9, 11, 13, 17, 19, 21}$ are not hard. I will try to do 11, 13, 17, 21, but you should really pay attention to 5.

27. (Integral test). $f(x)=\frac{1}{x(\ln x)^p}$ is continuous and positive on $[2,+\infty)$, and $f'(x)=-\frac{p+\ln x}{x^2(\ln x)^{p+1}}<0$ if $x>e^{-p}$, so that f is eventually decreasing and we can use the integral test.

When $p\neq 1$, $\int_2^{+\infty}\frac{1}{x(\ln x)^p}dx=\lim_{t\to\infty}\int_2^t\frac{1}{x(\ln x)^p}dx=\lim_{t\to\infty}\frac{(\ln x)^{1-p}}{1-p}\Big|_2^t$

$=\lim_{t\to\infty}\frac{(\ln t)^{1-p}}{1-p}-\frac{(\ln 2)^{1-p}}{1-p}$. Notice that $\lim_{t\to\infty}\ln t=\infty$, so this limit exists whenever $1-p<0 \iff p>1$.

When $p=1$ $\int_2^{+\infty}\frac{1}{x(\ln x)^p}dx=\lim_{t\to\infty}\int_2^t\frac{1}{x\ln x}dx=\lim_{t\to\infty}\ln(\ln x)\Big|_2^t$

$=\lim_{t\to\infty}\ln(\ln t)-\ln(\ln 2)=\infty$.

Combine the above two cases, the series $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converges for $p>1$ and diverges for $p\leq 1$.

29. (a) $f(x)=\frac{1}{x^2}$ is positive and continuous and $f'(x)=-\frac{2}{x^3}$ is negative for $x>0$, so the integral test applies.

$\sum_{n=1}^{\infty}\frac{1}{n^2}\approx S_{10}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2}\approx 1.549768$

$R_{10}\leq\int_{10}^{\infty}\frac{1}{x^2}dx=\lim_{t\to\infty}\int_{10}^t\frac{1}{x^2}dx=\lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_{10}^t=\lim_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{10}\right)=\frac{1}{10}$

So the error is at most 0.1.

(b) $S_{10}+\int_{11}^{\infty}\frac{1}{x^2}dx\leq S\leq S_{10}+\int_{10}^{\infty}\frac{1}{x^2}dx \Rightarrow S_{10}+\frac{1}{11}\leq S\leq S_{10}+\frac{1}{10}$

$\Rightarrow 1.549768+0.090909\leq S\leq 1.549768+0.1$

$\Rightarrow 1.640677\leq S\leq 1.649768$

So if we take the average of 1.640677 and 1.649768, namely $S\approx\frac{1}{2}(1.640677+1.649768)\approx 1.64522$, then the error ≤ 0.05 (obtained by taking the maximum of $1.649768-1.64522$ and $1.64522-1.640677$, rounded up).

(c) $R_n\leq\int_n^{\infty}\frac{1}{x^2}dx=\frac{1}{n}$, so $R_n<0.001$ if $\frac{1}{n}<0.001 \iff n>1000$

(I probably don't have time to talk about 29, so please read the solution carefully.)