Notes to Sections 2007. 1. 30

§6.5 P479

Idea about problems 5, 7, 13, 21 :

In all these 4 problems, the force is a function of distance, in which case you can always choose the distance as your variable to integrate with respect to. If you are moving an object from a to b, then the distance variable x varies from a to b. If the force can be expressed as a function of x, say $f(x)$, then the total work needed is $W=\int_a^b f(x)dx$. More details on page 472.

7. (a) Note that the natural length is 30cm, so if the spring is stretched from a length of 30 cm to 42 cm, the amount it is stretched is really from 0 m to 0.12 m. So we have

$$\int_0^{0.12} kx\,dx = 2 \Rightarrow \frac{1}{2}kx^2\Big|_0^{0.12} = 2 \Rightarrow 0.0072\,k = 2$$

$$\Rightarrow k = \frac{2}{0.0072} = \frac{2500}{9}\ N/m$$

Thus, the work needed to stretch the spring from 35 cm to 40 cm is

$$\int_{0.05}^{0.10} \frac{2500}{9}x\,dx = \frac{1250}{9}x^2\Big|_{0.05}^{0.10} = \frac{25}{24} \approx 1.04\,J$$

(b) $f(x)=kx$ so $30=\frac{2500}{9}x \Rightarrow x=\frac{270}{2500}=0.108\ m = 10.8\ cm$

13. The bucket is lifted from a height of 0 m to a height of 12 m. At a height of x meters $(0\le x\le 12)$, the mass of the rope is $0.8\cdot(12-x)$ kg and the mass of the water is $\frac{36}{12}\cdot(12-x)$ kg (reason: it's clear that the water leaks $\frac{36}{12}=3$ kg/m, so at a height of x meters, the water left in the bucket is $(36-3x)$ kg). The mass of the

bucket is 10kg, so the total mass is $0.8(12-x)+3(12-x)+10$
$=(55.6-3.8x)$ kg, and hence, the total force is $9.8\cdot(55.6-3.8x)$ N.
The work needed to lift the bucket Δx m through the i-th subinterval of $[0,12]$ is $9.8(55.6-3.8x_i^*)\Delta x$, so the total work is

$$W=\lim_{n\to\infty}\sum_{i=1}^{n} 9.8(55.6-3.8x_i^*)\Delta x=\int_0^{12} 9.8(55.6-3.8x)\,dx = \cdots$$

21. (a) $W=\int_a^b F(r)\,dr=\int_a^b G\frac{m_1m_2}{r^2}\,dr = Gm_1m_2\left(-\frac{1}{r}\right)\Big|_a^b = Gm_1m_2\left(\frac{1}{a}-\frac{1}{b}\right)$

(b) By part (a), $m_1=$ mass of earth $=5.98\times10^{24}$ kg
$m_2=$ mass of the satellite $=1000$ kg
$a=$ radius of earth $=6.37\times10^6$ m
$b=$ radius of earth + height of the satellite $=6.37\times10^6+1000\times10^3$
$=7.37\times10^6$ m (keep an eye on the unit of the given data)
Thus $W=6.67\times10^{-11}\times5.98\times10^{24}\times1000\times\left(\frac{1}{6.37\times10^6}-\frac{1}{7.37\times10^6}\right)\approx 8.50\times10^9$ J

§5.7. P 404.

23 $$\frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2}$$

Multiply both sides by $(x^2+1)(x^2+2)$ to get
$x^3+x^2+2x+1=(Ax+B)(x^2+2)+(Cx+D)(x^2+1)$
$=\cdots=(A+C)x^3+(B+D)x^2+(2A+C)x+(2B+D)$

Comparing coefficients gives us the following system of equations:

$A+C=1$ (1) $\qquad B+D=1$ (2)

$2A+C=2$ (3) $\qquad 2B+D=2$ (4)

Substracting equation (1) from (3) gives us $A=1$, so $C=0$.

Substracting equation (2) from (4) gives us $B=0$, so $D=1$.

Thus $I=\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int\left(\frac{x}{x^2+1}+\frac{1}{x^2+2}\right)dx$

For $\int \frac{x}{x^2+1}\,dx$, let $u=x^2+1$, so $du=2x\,dx$ and then

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u|+C_1 = \frac{1}{2}\ln(x^2+1)+C_1$$

For $\int \frac{1}{x^2+2}\,dx$, use the formula on page 403, with $a=\sqrt{2}$, we have

$$\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C_2$$

Thus, $I=\frac{1}{2}\ln(x^2+1)+\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+C$

27. By long division, $\frac{x^3+4}{x^2+4} = x+\frac{-4x+4}{x^2+4}$

$$\begin{array}{r} x+0 \\ x^2+4\,\overline{)\,x^3+0x^2+0x+4} \\ x^3+0x^2+4x \\ \hline -4x+4 \end{array}$$

Thus $\int \frac{x^3+4}{x^2+4}\,dx = \int\left(x+\frac{-4x+4}{x^2+4}\right)dx$

$$= \int x\,dx - 4\int \frac{x}{x^2+4}\,dx + 4\int \frac{1}{x^2+4}\,dx$$

$$= \frac{1}{2}x^2 - 4\cdot\frac{1}{2}\ln(x^2+4) + 4\cdot\frac{1}{2}\arctan\frac{x}{2}+C$$

$$= \frac{1}{2}x^2 - 2\ln(x^2+4) + 2\arctan\frac{x}{2}+C$$

29. Let $u=\sqrt{x}$ so $u^2=x$ and $dx=2u\,du$. Thus

$$I=\int_9^{16}\frac{\sqrt{x}}{x-4}\,dx = \int_3^4 \frac{u}{u^2-4}\,2u\,du = 2\int_3^4\frac{u^2}{u^2-4}\,du = 2\int_3^4\left(1+\frac{4}{u^2-4}\right)du \quad \left(\text{by long division}\right)$$

$$= 2+8\int_3^4\frac{du}{(u+2)(u-2)}$$

Multiply $\frac{1}{(u+2)(u-2)} = \frac{A}{u+2}+\frac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$.

Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $A=-\frac{1}{4}$ and $B=\frac{1}{4}$, so $\frac{1}{(u+2)(u-2)} = \frac{-1/4}{u+2}+\frac{1/4}{u-2}$ and the original integral becomes:

$$I = 2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du = 2+8\left(-\frac{1}{4}\ln|u+2|+\frac{1}{4}\ln|u-2|\right)\Big|_3^4$$
$$= 2+\left(2\ln|u-2|-2\ln|u+2|\right)\Big|_3^4 = 2+2\left(\ln\left|\frac{u+2}{u-2}\right|\right)\Big|_3^4$$
$$= 2+2\left(\ln\frac{2}{6}-\ln\frac{1}{5}\right) = 2+2\ln\frac{5}{3}$$

A few words about the exam:

Questions on the actual exam should resemble a subset of the posted problems, so if you have enough time, try to work on (as many as possible) old exam questions and also take a look at the daily and weekly homework problems, in particular, those problems you found difficult at the beginning. Maybe you can also work on the review problems at the end of each chapter (Mark has already posted solutions to these on the official course website.)

Please help me to organize the section on Thursday! If you, want me to explain any particular problem on the review section on Thursday, please email me by Wednesday evening (let's say the cut-off time is 9 pm). Problems from the textbook and old exams are both welcome. If I've got too many requests, I will pick some common questions to explain on Thursday.

Good luck to your midterm exam!