Notes to Section 2007. 1. 25

Standard procedure of calculating volumes :

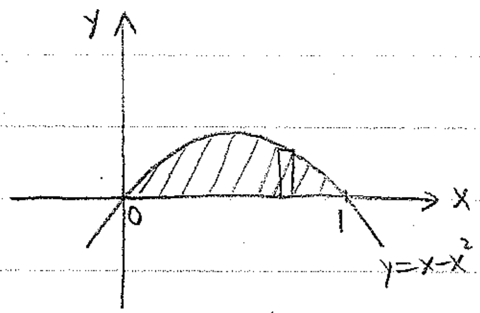
(1) determine the variable you want to integrate with respect to ;

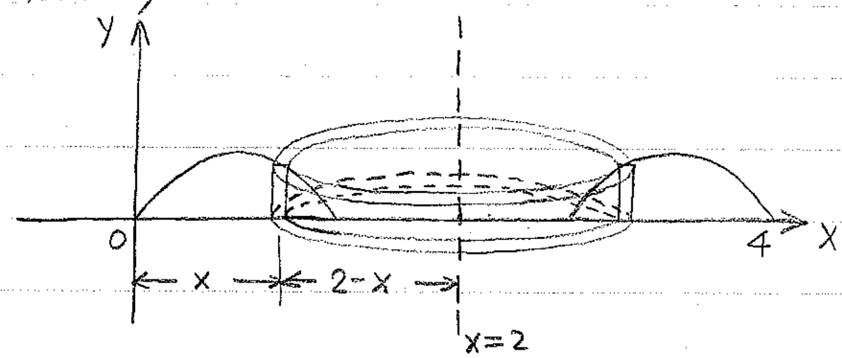
(2) calculate the area of a typical cross-section (or a cylindrical shell)

(3) integrate the area with respect to the variable you choose in (1).

§ 6.2 P460

53. (example of cylindrical shell method)





About the cylindrical shell :

circumference $= 2\pi\cdot(2-x)$

height $= x-x^2$

thickness $= dx$

so the volume : $V=\int_0^1 2\pi(2-x)\cdot(x-x^2)\cdot dx = 2\pi\int_0^1 (x^3-3x^2+2x)\,dx = \cdots$

The reason why the method of cylindrical shells is preferable to slicing :
If we were to use the "washer method", we would first have to locate the maximal point of $y=x-x^2$ using the method of Chapter 4. Then we would to have to solve the equation $y=x-x^2$ for x in terms of y to obtain the radii of the washer, which is difficult because it involves quadratic formula.

Procedure of calculating work done in moving an object:

(1) Cut the object into many pieces such that points in every single piece move (almost) the same distance;

(2) Calculate work done to every single piece by the formula work = force × distance;

(3) take the Riemann sum of work done for all pieces, and express the limit of the Riemann sum as a definite integral.

§ 6.5. P479

9. (a) The portion of the rope from x ft to $(x+\Delta x)$ ft below the top of the building weighs $\frac{1}{2}\Delta x$ lb and must be lifted x_i^* ft, where x_i^* is a sample point in the i-th subinterval $[x_{i-1}, x_i]$, so its contribution to the total work is $\frac{1}{2}x_i^*\Delta x$ ft-lb.

The total work is (n is the number of subintervals of length Δx)

$$W = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}x_i^*\Delta x = \int_0^{50}\frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_0^{50} = 625 \text{ ft-lb}.$$

Notice that the exact height of the building does not matter as long as it is more than 50 ft.

(b) When half of the rope is pulled to the top of the building, the work to lift the top half of the rope is:

$$W_1 = \int_0^{25}\frac{1}{2}x\,dx = \frac{1}{4}x^2\Big|_0^{25} = \frac{625}{4} \text{ ft-lb}$$

The bottom half of the rope is lifted 25 ft and the work needed to accomplish that is $W_2 = \frac{1}{2}\cdot 25\cdot 25 = \frac{625}{2}$ ft-lb

The total work done in pulling half the rope to the top of the building is

$$W = W_1 + W_2 = \frac{625}{4} + \frac{625}{2} = \frac{1875}{4} \text{ ft-lb}$$

11. The work needed to lift the cable is

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^* \Delta x = \int_0^{500} 2x\,dx = x^2\Big|_0^{500} = 250000 \text{ ft-lb}$$

The work needed to lift the coal is

$$800 \text{ lb} \cdot 500 \text{ ft} = 400000 \text{ ft-lb}$$

Thus, the total work required is $250000 + 400000 = 650000$ ft-lb.

15. A "slice" of water Δx m thick and lying at a depth of x_i^* m (where $0 \le x_i^* \le \frac{1}{2}$) has volume $(2 \times 1 \times \Delta x)$ m^3, a mass of $2000\,\Delta x$ kg, weighs about $9.8 \times 2000\Delta x = 19600\,\Delta x$ N, and thus requires about $19600\, x_i^* \Delta x$ J of work for its removal.

So $W = \lim_{n\to\infty} \sum_{i=1}^{n} 19600\, x_i^* \Delta x = \int_0^{\frac{1}{2}} 19600\, x\,dx = 9800\, x^2\Big|_0^{\frac{1}{2}} = 2450$ J

17. (a) A rectangular "slice" of water Δx m thick and lying x ft above the bottom has width x ft and volume $8x\,\Delta x$ m^3. It weighs about $9.8 \times 1000 \times 8x\Delta x$ N, and must be lifted $(5-x)$ m by the pump, so the work needed is about $9.8 \times 10^3 \times (5-x) \times (8x\,\Delta x)$ J. The total work is

$$W = \int_0^3 9.8 \times 10^3 \times (5-x) \times 8x\,dx = 9.8 \times 10^3 \int_0^3 (40x - 8x^2)\,dx$$
$$= 9.8 \times 10^3 \times \left(20x^2 - \frac{8}{3}x^3\right)\Big|_0^3 = 9.8 \times 10^3 \times (180 - 72) = 1058400 \text{ J}$$

(b) If only 4.7×10^5 J of work is done, then only the water above a certain level (call it h) will be pumped out. So we use the same formula as in part (a), except that the lower limit is unknown:

$$4.7 \times 10^5 = \int_h^3 (9.8 \times 10^3)(5-x)\, 8x\,dx$$

$\Leftrightarrow \cdots$ (do the calculation yourself) $\Leftrightarrow 2h^3 - 15h^2 + 45 = 0$

To find the solution, we plot $2h^3 - 15h^2 + 45$ between $h=0$ and $h=3$.

We see $h \approx 2.0$ m. So the depth of the water remaining is about 2.0 m.