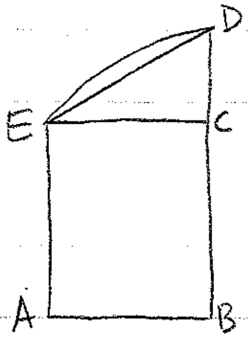
Notes to Section 2007.1.23

§5.9 P421

1 (d) For any n, we will have $L_n < T_n < I < M_n < R_n$

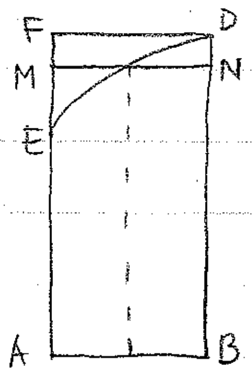
Reason: First of all, the function f is increasing which implies L_n is an underestimate and R_n is an overestimate.



Second, f is concave down which implies M_n is an overestimate and T_n is an underestimate.

So now we have $L_n < I < R_n$ and $T_n < I < M_n$

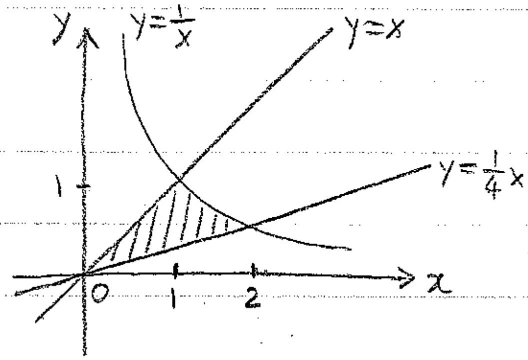
In order to compare L_n and T_n, take a close look at what happens on each small interval (as shown on the left). On such a small interval, left endpoint approximation measures the area of the rectangle ABCE while the trapezoidal rule measures the area of the trapezoid ABDE, therefore $L_n < T_n$.



In order to compare M_n and R_n, also look at a small interval (as shown on the left). The midpoint rule measures the area of the rectangle ABNM while the right endpoint approximation measures the area of the rectangle ABDF, therefore $M_n < R_n$.

In fact, for increasing functions, it's always true that $L_n < T_n < R_n$ and $L_n < M_n < R_n$.

§6.1 P446



15. First, find the coordinates of the intersection point

$\frac{1}{x}=x \Leftrightarrow x^2=1 \Leftrightarrow x=\pm 1$

$\frac{1}{x}=\frac{x}{4} \Leftrightarrow x^2=4 \Leftrightarrow x=\pm 2$

So for $x>0$, $y=x$ and $y=\frac{1}{x}$ intersect at $(1,1)$

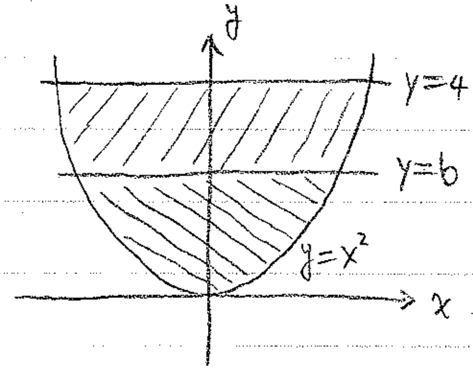
$y=\frac{1}{4}x$ and $y=\frac{1}{x}$ intersect at $(2,\frac{1}{2})$

The vertical line $x=1$ cuts the shaded area into two pieces, hence the area can be written as the sum of two integrals:

$$A=\int_0^1 (x-\tfrac{1}{4}x)\,dx+\int_1^2 (\tfrac{1}{x}-\tfrac{1}{4}x)\,dx = \cdots$$

23. If x = horizontal distance from left end of pool and $w=w(x)$ = width at x, then Simpson's rule with $n=8$ and $\Delta x=2$ gives: $\text{Area}=\int_0^{16} w(x)\,dx \approx \frac{2}{3}[0+4\cdot 6.2+2\cdot 7.2+4\cdot 6.8+2\cdot 5.6+4\cdot 5.0+2\cdot 4.8+0]=\frac{2}{3}\cdot 126.4\approx 84.3\ m^2$

Remark: don't forget the unit in an application problem.

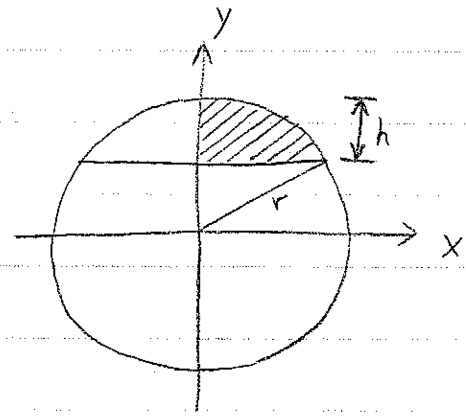


39. Since we need to divide the whole region by a horizontal line, a more convenient way is to integrate with respect to y. For any value of y, $y=x^2 \Leftrightarrow x=\pm\sqrt{y}$, so x varies between $-\sqrt{y}$ and $\sqrt{y}$. We are looking for a number b such that the two shaded regions have equal area, i.e.

$$\int_0^b [\sqrt{y}-(-\sqrt{y})]\,dy=\int_b^4 [\sqrt{y}-(-\sqrt{y})]\,dy \Rightarrow \cdots$$

§6.2. P457

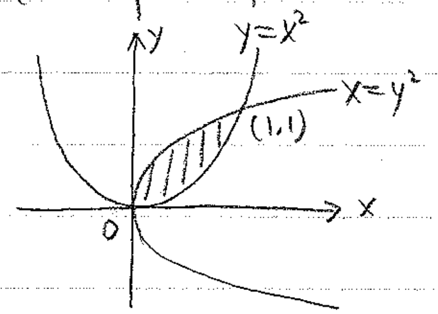
27. (example of disk-shape cross-section)

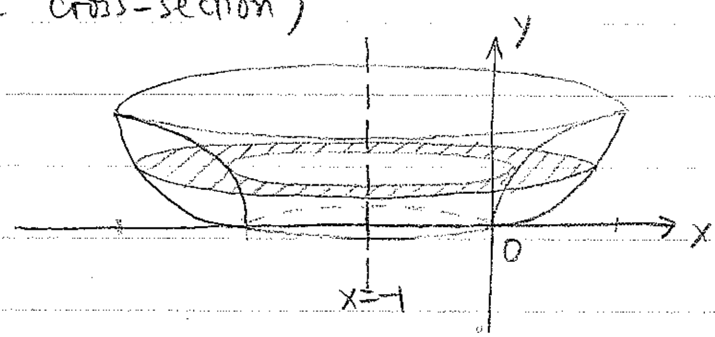


The solid cap of the sphere is obtained by rotating the shaded region about the y-axis. For convenience we choose the coordinate system as shown in the picture. The a typical cross-section at a distance y from the origin has a radius $\sqrt{r^2-y^2}$, hence $A(y)=\pi(r^2-y^2)$

Then $V=\int_{r-h}^{r} A(y)\,dy = \int_{r-h}^{r} \pi(r^2-y^2)\,dy = \cdots$

11. (example of washer-shape cross-section)





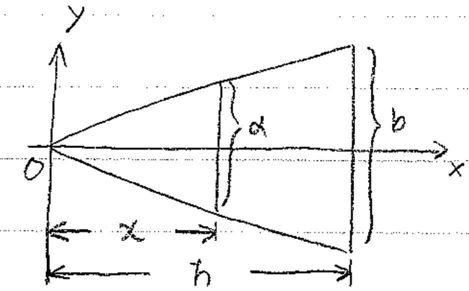
First solve the intersections of the two curves, which are $(0,0)$ and $(1,1)$. A typical cross-section is a washer. The outer radius is the distance from $x=-1$ to $x=\sqrt{y}$ (since $y=x^2 \Rightarrow x=\sqrt{y}$ for $x\geq 0$), and the inner radius is the distance from $x=-1$ to $x=y^2$. Therefore

$$A(y) = \pi(\text{outer radius})^2 - \pi(\text{inner radius})^2$$
$$= \pi\left[\sqrt{y}-(-1)\right]^2 - \pi\left[y^2-(-1)\right]^2$$

and the volume is $V=\int_0^1 A(y)\,dy = \cdots$

29 (non-rotational example).

As in example 8, for convenience, we can place the origin O at the vertex of the pyramid and the x-axis perpendicular to the base of the pyramid.



A typical cross-section is a rectangle with dimensions α and β.
We see from similar triangles that $\frac{\alpha/2}{b/2} = \frac{x}{h}$, so $\alpha = \frac{bx}{h}$.
Similarly, by replacing b by 2b, we get $\beta = \frac{2bx}{h}$.
So the area of a cross-section at a distance of x from the vertex is $A(x) = \frac{bx}{h} \cdot \frac{2bx}{h} = \frac{2b^2x^2}{h^2}$

And the volume is

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{2b^2x^2}{h^2}\,dx = \frac{2b^2}{h^2}\int_0^h x^2\,dx$$

$$= \frac{2b^2}{h^2}\left.\frac{x^3}{3}\right|_0^h = \frac{2b^2}{h^2}\cdot\frac{h^3}{3} = \frac{2}{3}b^2h.$$