Notes to Section on 2007, 1, 18

Midpoint Rule: $\int_a^b f(x)\,dx \approx M_n = \Delta x[f(\bar{x}_1)+f(\bar{x}_2)+\cdots+f(\bar{x}_n)]$

where $\Delta x = \frac{1}{n}(b-a)$ $\quad \bar{x}_i$ = midpoint of $[x_{i-1}, x_i]$

Error bound : Suppose $|f''(x)| \le K_2$ for $a \le x \le b$,

$|E_M| \le \frac{K_2(b-a)^3}{24n^2}$

(Remark: The error is the difference of the actual value of the integral and the approximation. It is a certain number whose absolute value does not exceed the error bound. So the error bound only tells you sort of the "worst" possible situation. In many cases, midpoint rule works better than the "worst" case.)

§ 5.9 P421 (We only consider Midpoint rule today)

1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$, so $x_0=0$, $x_1=2$, $x_2=4$

$L_2 = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 0.5\times 2 + 2.5\times 2 = 6$

$R_2 = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2.5\times 2 + 3.5\times 2 = 12$

$M_2 = f(\bar{x}_1)\cdot 2 + f(\bar{x}_2)\cdot 2 = 2f(1) + 2f(3) \approx 2(1.6+3.2) = 9.6$

(b) L_2 is an underestimate, since if you draw the small rectangles, the area of the small rectangles is less than the area under the curve. R_2 is an overestimate, since similarly we can draw some larger rectangles, and the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it's fairly closed to I.

Remark: it's a fact that if $f(x)$ is concave down on $[a,b]$, then the Midpoint rule is an overestimate of $\int_a^b f(x)\,dx$. The reason is clear if you look at figure 5 on page 915 carefully. The Midpoint Rule gives the area of the trapezoid ABCD, while the actual integral is the area of the region AQPRD, which is smaller than the trapezoid ABCD.

5. $f(x) = x^2 \sin x$, $\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{8} = \frac{\pi}{8}$

$M_8 = \frac{\pi}{8}\left[f(\frac{\pi}{16}) + f(\frac{3\pi}{16}) + f(\frac{5\pi}{16}) + \cdots + f(\frac{15\pi}{16})\right] \approx \cdots$ (6 decimals)

Actual: $\int_0^\pi x^2 \sin x\,dx = -\int_0^\pi x^2 d(\cos x) = -x^2\cos x\Big|_0^\pi + \int_0^\pi \cos x \cdot d(x^2)$

$= -[\pi^2(-1) - 0] + 2\int_0^\pi x\cos x\,dy = \pi^2 + 2\int_0^\pi x\,d(\sin x)$

$= \pi^2 + 2x\sin x\Big|_0^\pi - 2\int_0^\pi \sin x\,dx = \pi^2 + 2(0-0) - 2(-\cos x)\Big|_0^\pi$

$= \pi^2 + 2(-1-1) = \pi^2 - 4 \approx \cdots$ (6 decimals)

Error: $E_M = \text{actual} - M_8 \approx \cdots$ (6 decimals)

15. $f(y) = \frac{1}{1+y^5}$, $\Delta y = \frac{3-0}{6} = \frac{1}{2}$

$M_6 = \frac{1}{2}\left[f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4}) + f(\frac{9}{4}) + f(\frac{11}{4})\right] \approx \cdots$

17. (a) $f(x) = e^{-x^2}$, $\Delta x = \frac{2-0}{10} = 0.2$

$M_{10} = \frac{1}{5}\left[f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.9)\right] \approx \cdots$

(b) $f(x) = e^{-x^2}$, $\Rightarrow f'(x) = -2xe^{-x^2}$, $f''(x) = (4x^2-2)e^{-x^2}$

In order to find the extrema of $f''(x)$, we need $f'''(x) = 4x(3-2x^2)e^{-x^2}$.

Special points we need to consider:

endpoints: $x=0$ or $x=2$, critical points: $f'''(x)=0$, $x=0$ or $\pm\sqrt{\frac{3}{2}}$

So we need to consider $x=0$, $x=\sqrt{\frac{3}{2}}$, $x=2$

$|f''(0)|=2$, $|f''(\sqrt{\frac{3}{2}})| \approx 0.8925$, $|f''(2)| \approx 0.2564$. Thus, taking $K_2=2$

By the error bound formula, $|E_M| \leq \frac{2(2-0)^3}{24 \cdot 10^2} = 0.00\overline{6}$

(c). We have $K_2=2$ in part (b). We want the error bound to be less than or equal to 0.00001 so that we can make sure the error is controlled within the allowed interval.

So we want: $\frac{2(2-0)^3}{24n^2} \leq 0.00001 \Rightarrow n^2 \geq \frac{2}{3} \cdot 10^5$

$\Rightarrow n \geq \sqrt{\frac{2}{3} \cdot 10^5} \approx 258.2$ (since n is positive)

So we have to choose n to be at least 259 (since n is an integer)

19. Let's only do (c)

$f(x)=e^x \Rightarrow f'(x)=e^x$ and $f''(x)=e^x$ which is increasing on $[0,1]$

so the extrema of $f''(x)$ occur at endpoints hence $|f''(x)| \leq e$ on $[0,1]$

i.e. $K_2=e$ We want $\frac{K_2(b-a)^3}{24n^2} \leq 0.00001$

i.e. $\frac{2 \cdot 1^3}{24n^2} \leq 0.00001 \Rightarrow n \geq 106.4$ hence n should be at least 107.

31. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint rule with $n=8/2=4$ and $\Delta x = (3.2-0)/4 = 0.8$, so $\int_0^{3.2} f(x)\,dx \approx M_4 = 0.8[f(0.4)+f(1.2)+f(2.0)+f(2.8)]$

$= \cdots$

(b) $-4 \leq f''(x) \leq 1 \Rightarrow |f''(x)| \leq 4$ hence $K_2=4$, so

$|E_M| \leq \frac{4 \cdot (3.2-0)^3}{24 \cdot 4^2} = \cdots$