Partial Solutions to Exercises 2007.1.16

§5.7. P404

1. $\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot (\sin x \, dx) = \int (1-\cos^2 x) \cdot \cos^2 x \cdot (\sin x \, dx)$

let $u = \cos x$, then $du = -\sin x \, dx$,

the above integral becomes

$\int (1-u^2) \cdot u^2 \cdot (-du) = \int (u^4 - u^2) \, du = \cdots$

(try to finish off the solution yourself)

5. $\int_0^{2\pi} \cos^2(6\theta) \, d\theta = \int_0^{2\pi} \frac{1}{2} [1 + \cos(12\theta)] \, d\theta = \frac{1}{2} [\theta + \frac{1}{12} \sin(12\theta)] \Big|_0^{2\pi}$

$= \cdots$ (finish off the solution yourself)

7. remember $du = \sec x \cdot \tan x \cdot dx$, so

$\int \tan^3 x \cdot \sec x \cdot dx = \int \tan^2 x \cdot (\tan x \cdot \sec x \cdot dx)$

$= \int (\sec^2 x - 1) \cdot (\tan x \cdot \sec x \cdot dx) = \int (u^2 - 1) \, du = \cdots$

(finish off the solution yourself)

11. $x = 2\tan\theta$ then $dx = 2\sec^2\theta \, d\theta$

and $\sqrt{x^2+4} = \sqrt{4\tan^2\theta + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta}$

Since $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have $\sec\theta > 0$, and

$\sqrt{x^2+4} = \sqrt{4\sec^2\theta} = 2\sec\theta$

Thus, substitution gives:

$\int \frac{1}{x^2\sqrt{x^2+4}} \, dx = \int \frac{1}{4\tan^2\theta \cdot 2\sec\theta} \cdot 2\sec^2\theta \, d\theta = \frac{1}{4} \int \frac{\sec\theta}{\tan^2\theta} \, d\theta$

$= \frac{1}{4} \int \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta} \, d\theta = \frac{1}{4} \int \frac{\cos\theta}{\sin^2\theta} \, d\theta = \frac{1}{4} \int \frac{1}{u^2} \, du$

by setting $u = \sin\theta$ (finish off the solution yourself)

13. Let $t = \sec\theta$, then $dt = \sec\theta \tan\theta \, d\theta$

New integral limits: $t = \sqrt{2} \Rightarrow \theta = \frac{\pi}{4}$

$t = 2 \Rightarrow \theta = \frac{\pi}{3}$

(Reason: $t = \sqrt{2} \Rightarrow \sec\theta = \sqrt{2} \Rightarrow \cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$t = 2 \Rightarrow \theta = \frac{\pi}{3}$ is similar)

$\sqrt{t^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$ since

$\tan\theta > 0$ when θ is between $\frac{\pi}{4}$ and $\frac{\pi}{3}$

The substitution gives:

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\, dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3\theta \cdot \tan\theta} \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta}\, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \cdots$$ (finish off the solution yourself)

31 Completing the square: $x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1$

(principle: add and substract the square of half of the coefficient of x)

$$= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

Then the integral $= \int \frac{1}{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}\, dx$

Let $u = x + \frac{1}{2}$, then $du = dx$

the integral $= \int \frac{1}{u^2 + (\frac{\sqrt{3}}{2})^2}\, du = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u}{(\frac{\sqrt{3}}{2})}\right) + C$

(by formula on page 403)

$$= \frac{2}{\sqrt{3}} \tan^{-1}\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C = \frac{2}{\sqrt{3}} \tan^{-1}\frac{2x+1}{\sqrt{3}} + C$$

Additional problem: $\int e^x\sqrt{1-e^{2x}}\,dx$

Let $u=e^x$, then $du=e^x dx$, so

the integral $=\int\sqrt{1-(e^x)^2}\,(e^x dx)=\int\sqrt{1-u^2}\,du$

Let $u=\sin\theta$ $(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2})$, we have $\cos\theta\ge 0$ for $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

then $\sqrt{1-u^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}=\cos\theta$, $du=\cos\theta\,d\theta$

the integral $=\int\cos\theta\cdot\cos\theta\,d\theta=\int\cos^2\theta\,d\theta=\int\frac{1}{2}[1+\cos(2\theta)]\,d\theta$

$=\frac{1}{2}[\theta+\frac{1}{2}\sin(2\theta)]+C$

$=\frac{1}{2}[\arcsin u+\frac{1}{2}\sin(2\arcsin u)]+C$

$=\frac{1}{2}[\arcsin e^x+\frac{1}{2}\sin(2\arcsin e^x)]+C$

The <u>optional</u> simplification step:

$\sin(2\arcsin e^x)=\sin(2\theta)=2\sin\theta\cos\theta$

$=2\sin\theta\sqrt{\cos^2\theta}=2\sin\theta\sqrt{1-\sin^2\theta}$

$=2u\sqrt{1-u^2}=2e^x\sqrt{1-(e^x)^2}=2e^x\sqrt{1-e^{2x}}$

So the original integral $=\frac{1}{2}(\arcsin e^x+e^x\sqrt{1-e^{2x}})+C$

Tips Having the following tables memorized is very helpful:

Table of Indefinite integrals on page 369 of the text,

Table of trigonometric substitutions on the handout,

Some frequently used identities involving trigonometric functions.

(you can find some of them on the reference pages in the text)