SOLUTIONS TO HANDOUT EXAMPLES

Problem 1

Problem. Evaluate $\int x^5 \cos x^3 dx$. $\square$

Solution. Use substitution first. I want to reserve the letter u in the next step so I will use t here.

Let $t = x^3$, then $dt = 3x^2 dx$, so $x^2 dx = \frac{1}{3} dt$. Therefore,

$$\int x^5 \cos x^3 dx = \int x^3 \cos x^3 \cdot x^2 dx = \int t \cos t \cdot \frac{1}{3} dt = \frac{1}{3} \int t \cos t dt.$$

Next we use integration by parts. Let $u = t$ and $dv = \cos t dt$, then $du = dt$ and $v = \sin t$. Therefore,

$$\int t \cos t dt = t \sin t - \int \sin t dt = t \sin t - (-\cos t) + C' = t \sin t + \cos t + C'.$$

Hence

$$\begin{aligned} \int x^5 \cos x^3 dx &= \frac{1}{3}(t \sin t + \cos t + C') \\ &= \frac{1}{3} t \sin t + \frac{1}{3} \cos t + C \\ &= \frac{1}{3} x^3 \sin x^3 + \frac{1}{3} \cos x^3 + C, \end{aligned}$$

where C is any constant. $\square$

Problem 2

Problem. Evaluate $\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3 \theta d\theta$. $\square$

Solution. Note that the function $\tan^3 x$ is an odd function because

$$\tan^3(-x) = (-\tan x)^3 = -\tan^3 x.$$

And the inteval is symmetric about the origin. So

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3 \theta d\theta = 0.$$

$\square$

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Problem 3

Problem. A particle is moving along a line. The acceleration function (in m/s^2) is given by $a(t) = 2t + 3$ and the initial velocity (in m/s) $v(0) = -4$. Find the displacement and the distance traveled during the time interval $0 \leqslant t \leqslant 3$. □

Solution. Since $a(t) = 2t + 3$, the velocity function is an antiderivative of this function. Integrate it and we get $v(t) = t^2 + 3t + C$. So $v(0) = 0 + 0 + C = C$. However we know that $v(0) = -4$, so $C = -4$, therefore $v(t) = t^2 + 3t - 4$.
The displacement is the integral of the velocity function, that is,

$$\int_0^3 (t^2 + 3t - 4)dt = (\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t)\Big]_0^3 = \frac{21}{2}.$$

The distance traveled by the partical is the integral of the absolute value of the velocity function, that is,

$$\int_0^3 |t^2 + 3t - 4|dt.$$

We have to consider the sign of the velocity function. Note that $x^2 + 3x - 4 = (x + 4)(x - 1)$. So $x^2 + 3x - 4 = 0$ has solutions $x = -4$ and $x = 1$. It's not hard to see that $x^2 + 3x - 4$ is positive on intervals $(-\infty, -4)$ and $(1, \infty)$, and is negative on the interval $(-4, 1)$. Therefore we have to break up the integral at $t = 1$.

$$\begin{aligned} \int_0^3 |t^2 + 3t - 4|dt &= \int_0^1 |t^2 + 3t - 4|dt + \int_1^3 |t^2 + 3t - 4|dt \\ &= \int_0^1 [-(t^2 + 3t - 4)]dt + \int_1^3 (t^2 + 3t - 4)dt \\ &= (-\frac{1}{3}t^3 - \frac{3}{2}t^2 + 4t)\Big]_0^1 + (\frac{1}{3}t^3 + \frac{3}{2}t^2 - 4t)\Big]_1^3 \\ &= \frac{13}{6} + \frac{38}{3} \\ &= \frac{89}{6}. \end{aligned}$$

□

Problem 4

Problem. Use the limit of a Riemann sum to evaluate the integral $\displaystyle\int_{-1}^3 x^2 dx.$ □

Solution. In this problem, $f(x) = x^2$ and $\Delta x = \frac{3-(-1)}{n} = \frac{4}{n}$. We also have

$$x_i = x_0 + i\Delta x = -1 + \frac{4i}{n}.$$

Therefore by the Riemann sum formula for the right endpoints rule, we have

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} x_i^2 \Delta x \\
&= \sum_{i=1}^{n} (-1+\frac{4i}{n})^2 \frac{4}{n} \\
&= \frac{4}{n}\sum_{i=1}^{n}\left(1-2\cdot\frac{4i}{n}+(\frac{4i}{n})^2\right) \\
&= \frac{4}{n}\left(\sum_{i=1}^{n}1-\frac{8}{n}\sum_{i=1}^{n}i+\frac{16}{n^2}\sum_{i=1}^{n}i^2\right) \\
&= \frac{4}{n}\left(n-\frac{8}{n}\cdot\frac{1}{2}n(n+1)+\frac{16}{n^2}\cdot\frac{1}{6}n(n+1)(2n+1)\right) \\
&= 4-16\cdot\frac{n+1}{n}+\frac{32}{3}\cdot\frac{(n+1)(2n+1)}{n^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{-1}^{3} x^2 dx &= \lim_{n\to\infty}\left(4-16\cdot\frac{n+1}{n}+\frac{32}{3}\cdot\frac{(n+1)(2n+1)}{n^2}\right) \\
&= 4-16\cdot 1+\frac{32}{3}\cdot 2 \\
&= \frac{28}{3}.
\end{aligned}$$

□

Problem 5

Problem. Differentiate the function $y = \int_{\sin x}^{\cos x} (1+v^2)^{10} dv.$ □

Solution. We want to use fundamental theorem, so we need to break the integral into two parts.

$$\begin{aligned}
y &= \int_{\sin x}^{\cos x} (1+v^2)^{10} dv \\
&= \int_{\sin x}^{0} (1+v^2)^{10} dv + \int_{0}^{\cos x} (1+v^2)^{10} dv \\
&= -\int_{0}^{\sin x} (1+v^2)^{10} dv + \int_{0}^{\cos x} (1+v^2)^{10} dv
\end{aligned}$$

By FTC1 and chain rule, we have

$$\frac{dy}{dx} = -(1+\sin^2 x)^{10}\cdot\cos x + (1+\cos^2 x)^{10}\cdot(-\sin x).$$

□

Problem 6

Problem. A right circular cone is inscribed in a sphere of radius R. Find the largest possible volume of such a cone. □

Solution. Let's denote the radius of the base of the cone by r and the height by h, then the volume is

$$V = \frac{1}{3}\pi r^2 h.$$

From the picture (refer to figure 1 on the last page) we see a right-angled triangle in which the legs of the right angle are $h - R$ and r while the hypotenus is R. So we have a relation between r and h, that is

$$(h - R)2 + r^2 = R^2.$$

In order to eliminate r in the function, we have

$$r^2 = R^2 - (h - R)^2 = 2Rh - h^2.$$

Therefore

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(2Rh - h^2)h = \frac{1}{3}\pi(2Rh^2 - h^3).$$

The domain of the function is $0 < h < 2R$. Compute the critical points by differentiating the function:

$$V' = \frac{1}{3}\pi(4Rh - 3h^2) = \frac{1}{3}\pi h(4R - 3h).$$

So $V' = 0$ gives $h = 0$ or $h = \frac{4}{3}R$. However, $h = 0$ is not in domain. So the only critical point in domain is $h = \frac{4}{3}R$. Now we want to check this is the point where the function achieves its maximum by 1st derivative test.

On the interval $(0, \frac{4}{3}R)$, we have $h > 0$ and $4R - 3h > 0$, so $V' = \frac{1}{3}\pi h(4R - 3h) > 0$. While on the interval $(\frac{4}{3}R, 2R)$ we have $h > 0$ and $4R - 3h < 0$, so $V' = \frac{1}{3}\pi h(4R - 3h) < 0$. By 1st derivative test, the original function achieves its maximum at $h = \frac{4}{3}R$, which is

$$V\left(\frac{4}{3}h\right) = \frac{1}{3}\pi\left(2R(\frac{4}{3}R)^2 - (\frac{4}{3}R)^3\right) = \frac{32}{81}\pi R^3.$$

□

Problem 7

Problem. A 15-foot ladder is resting against the wall. The bottom is initially 10 feet away from the wall and is being pushed towards the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall 12 seconds after we start pushing? □

Solution. Refer to figure 2 on the last page. Let x be the distance between the bottom of the ladder and the corner and let y be the distance between the top of the ladder and the corner. Then x and y are related by the equation

$$x^2 + y^2 = 15^2.$$

Differentiate the equation with respect to t and we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0.$$

Hence we get

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

From the problem we know

$$\frac{dx}{dt} = -\frac{1}{4}.$$

And at the end of 12 seconds,

$$x = 10 - \frac{1}{4} \cdot 12 = 7,$$

and from the equation $x^2 + y^2 = 15^2$ we can get

$$y = \sqrt{15^2 - x^2} = \sqrt{15^2 - 7^2} = \sqrt{176} = 4\sqrt{11}.$$

Plug in these values and we have

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{7}{4\sqrt{11}} \cdot (-\frac{1}{4}) = \frac{7}{16\sqrt{11}}\text{ft/sec.}$$

The conclusion is: the top of the ladder is moving up the wall at the speed of $\frac{7}{16\sqrt{11}}$ ft/sec 12 seconds after we start pushing. □

Final Remark

Please contact me if you have any questions or find any mistakes in these solutions. Good luck to your finals!

figure 1

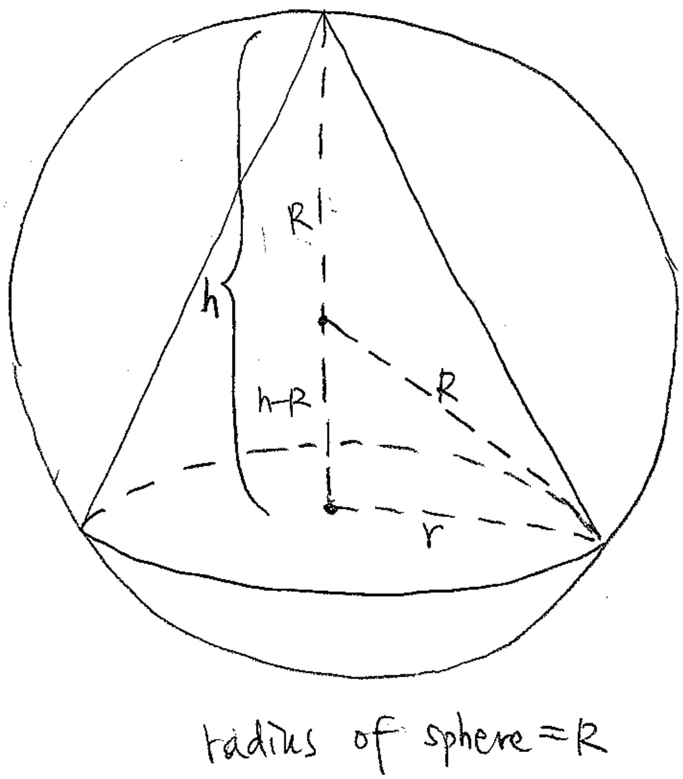

figure 2