Section Notes

12/04/07

Recall substitution:

(indefinite integral) $\int f(g(x))g'(x)\,dx = \int f(u)\,du$, where $u = g(x)$

(definite integral) $\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Integration by parts:

(indefinite integral) $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

OR $\int u\,dv = uv - \int v\,du$

(definite integral) $\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx$

Sometimes we need to combine the two big tools together with properties of integrals to compute.

$\boxed{1}$ Examples of substitution: P392

9. $\int (3x-2)^{20}\,dx$

$u = 3x-2 \Rightarrow du = 3dx \Rightarrow dx = \frac{1}{3}du$

$\int (3x-2)^{20}\,dx = \int u^{20}\cdot\frac{1}{3}du = \frac{1}{3}\int u^{20}du = \frac{1}{3}\frac{1}{21}u^{21} + C = \frac{1}{63}(3x-2)^{21} + C$

17. $\int \sin\pi t\,dt$

$u = \pi t \Rightarrow du = \pi\,dt \Rightarrow dt = \frac{1}{\pi}du$

$\int \sin\pi t\,dt = \int \sin u\cdot\frac{1}{\pi}du = \frac{1}{\pi}\int \sin u\,du = \frac{1}{\pi}(-\cos u) + C = -\frac{1}{\pi}\cos\pi t + C$

21. $\int \cos\theta\,\sin^6\theta\,d\theta$

$u = \sin\theta \Rightarrow du = \cos\theta\,d\theta$

$\int \cos\theta\sin^6\theta\,d\theta = \int u^6\,du = \frac{1}{7}u^7 + C = \frac{1}{7}\sin^7\theta + C$

25. $\int \frac{dx}{\sqrt{1-x^2}\sin^{-1}x}$

$u = \sin^{-1}x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$

$\int \frac{dx}{\sqrt{1-x^2}\sin^{-1}x} = \int \frac{1}{u}du = \ln|u| + C = \ln|\sin^{-1}x| + C$

33. $\int \frac{1+x}{1+x^2}dx$

$\int \frac{1+x}{1+x^2}dx = \int \frac{1}{1+x^2}dx + \int \frac{x}{1+x^2}dx$

1st term: $\int \frac{1}{1+x^2}dx = \arctan x + C_1$

2nd term: $\int \frac{x}{1+x^2}dx$

$u = 1+x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

$\int \frac{x}{1+x^2}dx = \int \frac{1}{u}\cdot\frac{1}{2}du = \frac{1}{2}\int\frac{1}{u}du = \frac{1}{2}\ln|u| + C_2 = \frac{1}{2}\ln(x^2+1) + C_2$ (since $x^2+1>0$)

so $\int \frac{1+x}{1+x^2}dx = \arctan x + \frac{1}{2}\ln(x^2+1) + C$

45. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

$u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du$

when $x=1$, $u=1$; when $x=4$, $u=2$

$\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_1^2 e^u \cdot 2du = 2\int_1^2 e^u du = 2e^u\Big|_1^2 = 2(e^2-e)$

47. $\int_1^2 x\sqrt{x-1}\,dx$

$u = x-1 \Rightarrow du = dx$

when $x=1$, $u=0$; when $x=2$, $u=1$

$\int_1^2 x\sqrt{x-1}\,dx = \int_0^1 (u+1)\sqrt{u}\,du = \int_0^1 (u^{\frac{3}{2}} + u^{\frac{1}{2}})du = \left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right)\Big|_0^1$

$= \left(\frac{2}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}$

64. $\int_0^3 x f(x^2)\,dx$

$u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}du$

when $x=0$, $u=0$; when $x=3$, $u=9$

$\int_0^3 x f(x^2)\,dx = \int_0^9 f(u)\cdot\frac{1}{2}du = \frac{1}{2}\int_0^9 f(u)du = \frac{1}{2}\cdot 4 = 2$

[2] Examples of integral by parts, P398

5. $\int re^{\frac{r}{2}}dr$

$u=r$, $dv=e^{\frac{r}{2}}dr$

$\Rightarrow du=dr$, $\int e^{\frac{r}{2}}dr=\int e^{u}\cdot 2du$ (substitution: $u=\frac{r}{2}\Rightarrow dr=2du$)

$=2\int e^{u}du=2e^{u}+C=2e^{\frac{r}{2}}+C$

so choose $v=2e^{\frac{r}{2}}$

$\int re^{\frac{r}{2}}dr=r\cdot 2e^{\frac{r}{2}}-\int 2e^{\frac{r}{2}}\cdot dr$

$=2re^{\frac{r}{2}}-2\int e^{\frac{r}{2}}dr$

$=2re^{\frac{r}{2}}-2\int e^{u}\cdot 2du$ (substitution: $u=\frac{r}{2}\Rightarrow dr=2du$)

$=2re^{\frac{r}{2}}-4\int e^{u}du$

$=2re^{\frac{r}{2}}-4e^{u}+C=2re^{\frac{r}{2}}-4e^{\frac{r}{2}}+C$

7. $\int x^2\sin\pi x\,dx$

$u=x^2$ $dv=\sin\pi x\,dx$

$\Rightarrow du=2x\,dx$, $\int \sin\pi x\,dx=\int \sin u\cdot\frac{1}{\pi}du$ (Substitution: $u=\pi x\Rightarrow dx=\frac{1}{\pi}du$)

$=\frac{1}{\pi}\int\sin u\,du=\frac{1}{\pi}(-\cos u)+C=-\frac{1}{\pi}\cos\pi x+C$

so choose $v=-\frac{1}{\pi}\cos\pi x$

$\int x^2\sin\pi x\,dx=x^2\cdot(-\frac{1}{\pi}\cos\pi x)-\int(-\frac{1}{\pi}\cos\pi x)\cdot(2x\,dx)$

$=-\frac{1}{\pi}x^2\cos\pi x+\frac{2}{\pi}\int x\cos\pi x\,dx$

(Integration by parts again!)

$u=x$, $dv=\cos\pi x\,dx$

$\Rightarrow du=dx$, $\int\cos\pi x\,dx=\int\cos u\cdot\frac{1}{\pi}du$ (substitution: $u=\pi x\Rightarrow dx=\frac{1}{\pi}du$)

$=\frac{1}{\pi}\int\cos u\,du=\frac{1}{\pi}\sin u+C=\frac{1}{\pi}\sin\pi x+C$

so choose $v=\frac{1}{\pi}\sin\pi x$

$\int x\cos\pi x\,dx=x\cdot\frac{1}{\pi}\sin\pi x-\int\frac{1}{\pi}\sin\pi x\,dx$

$=\frac{1}{\pi}x\sin\pi x-\frac{1}{\pi}\int\sin\pi x\,dx$

$=\frac{1}{\pi}x\sin\pi x-\frac{1}{\pi}\int\sin u\cdot\frac{1}{\pi}du$ (substitution: $u=\pi x\Rightarrow dx=\frac{1}{\pi}du$)

$$= \frac{1}{\pi} x \sin\pi x - \frac{1}{\pi^2} \int \sin u\, du$$
$$= \frac{1}{\pi} x \sin\pi x - \frac{1}{\pi^2} (-\cos u) + C$$
$$= \frac{1}{\pi} x \sin\pi x + \frac{1}{\pi^2} \cos\pi x + C$$

So $\int x^2 \sin\pi x\, dx = -\frac{1}{\pi} x^2 \cos\pi x + \frac{2}{\pi} \cdot \int x \cos\pi x\, dx$

$$= -\frac{1}{\pi} x^2 \cos\pi x + \frac{2}{\pi}\left(\frac{1}{\pi} x \sin\pi x + \frac{1}{\pi^2}\cos\pi x + C \right)$$
$$= -\frac{1}{\pi} x^2 \cos\pi x + \frac{2}{\pi^2} x \sin\pi x + \frac{2}{\pi^3} \cos\pi x + \frac{2}{\pi} C$$
$$= -\frac{1}{\pi} x^2 \cos\pi x + \frac{2}{\pi^2} x \sin\pi x + \frac{2}{\pi^3} \cos\pi x + C'$$

17. $\int_1^2 \frac{\ln x}{x^2} dx$

$u = \ln x, \quad dv = \frac{1}{x^2} dx$

$\Rightarrow du = \frac{1}{x} dx, \quad v = -\frac{1}{x}$

$$\int_1^2 \frac{\ln x}{x^2} dx = -\frac{1}{x}\ln x \Big|_1^2 - \int_1^2 \left(-\frac{1}{x}\right) \cdot \frac{1}{x} dx$$
$$= -\frac{1}{x}\ln x \Big|_1^2 + \int_1^2 \frac{1}{x^2} dx = -\frac{1}{2}\ln 2 + \left(-\frac{1}{x}\right)\Big|_1^2 = -\frac{1}{2}\ln 2 + \frac{1}{2}$$

19. $\int_0^1 \frac{y}{e^{2y}} dy$

$u = y \qquad dv = \frac{1}{e^{2y}} dy = e^{-2y} dy$

$\Rightarrow du = dy \qquad \int e^{-2y} dy = \int e^u \left(-\frac{1}{2} du\right)$ (substitution. $u = -2y \Rightarrow dy = -\frac{1}{2} du$)

$$= -\frac{1}{2}\int e^u du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-2y} + C$$

so choose $v = -\frac{1}{2} e^{-2y}$

$$\int_0^1 \frac{y}{e^{2y}} dy = y \cdot \left(-\frac{1}{2} e^{-2y}\right)\Big|_0^1 - \int_0^1 \left(-\frac{1}{2} e^{-2y}\right) dy$$
$$= -\frac{1}{2} y e^{-2y}\Big|_0^1 + \frac{1}{2}\int_0^1 e^{-2y} dy$$
$$= -\frac{1}{2} e^{-2} + \frac{1}{2}\int_0^{-2} e^u \cdot \left(-\frac{1}{2} du\right)$$ (substitution. $u = -2y \Rightarrow dy = -\frac{1}{2} du$)
$$= -\frac{1}{2} e^{-2} - \frac{1}{4}\int_0^{-2} e^u du$$
$$= -\frac{1}{2} e^{-2} + \frac{1}{4}\int_{-2}^0 e^u du$$
$$= -\frac{1}{2} e^{-2} + \frac{1}{4} e^u\Big|_{-2}^0$$
$$= -\frac{1}{2} e^{-2} + \frac{1}{4}(1 - e^{-2})$$
$$= -\frac{3}{4} e^{-2} + \frac{1}{4}$$

21 $\int_0^{\frac{1}{2}} \sin^{-1}x\,dx$

$u = \sin^{-1}x, \quad dv = dx$

$\Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx, \quad v = x$

$\int_0^{\frac{1}{2}} \sin^{-1}x\,dx = x\sin^{-1}x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} x \frac{1}{\sqrt{1-x^2}}dx$

$= \frac{1}{2}\cdot\frac{\pi}{6} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx = \frac{\pi}{12} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx$

Need substitution:

$u = 1-x^2 \Rightarrow du = -2x\,dx \Rightarrow x\,dx = -\frac{1}{2}du, \quad x=0 \Rightarrow u=1; \quad x=\frac{1}{2} \Rightarrow u=\frac{3}{4}$

So $\int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx = \int_1^{\frac{3}{4}} \frac{1}{\sqrt{u}}\cdot\left(-\frac{1}{2}du\right) = \frac{1}{2}\int_{\frac{3}{4}}^1 u^{-\frac{1}{2}}du$

$= \frac{1}{2}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Big|_{\frac{3}{4}}^1 = u^{\frac{1}{2}}\Big|_{\frac{3}{4}}^1 = 1-\frac{\sqrt{3}}{2}$

So $\int_0^{\frac{1}{2}} \sin^{-1}x\,dx = \frac{\pi}{12} - \left(1-\frac{\sqrt{3}}{2}\right) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$

23. $\int_1^2 (\ln x)^2 dx$

$u = (\ln x)^2, \quad dv = dx$

$\Rightarrow du = 2\ln x \cdot \frac{1}{x}dx \quad v = x$

$\int_1^2 (\ln x)^2 dx = x(\ln x)^2\Big|_1^2 - \int_1^2 x\cdot 2\ln x\cdot\frac{1}{x}dx = 2(\ln 2)^2 - 2\int_1^2 \ln x\,dx$

use integration by parts again to compute $\int_1^2 \ln x\,dx$

$u = \ln x \quad dv = dx$

$\Rightarrow du = \frac{1}{x}dx \quad v = x$

$\int_1^2 \ln x\,dx = x\ln x\Big|_1^2 - \int_1^2 x\cdot\frac{1}{x}dx = 2\ln 2 - \int_1^2 1\,dx = 2\ln 2 - 1$

So $\int_1^2 (\ln x)^2 dx = 2(\ln 2)^2 - 2\int_1^2 \ln x\,dx$

$= 2(\ln 2)^2 - 2\cdot(2\ln 2 - 1) = 2(\ln 2)^2 - 4\ln 2 + 2$

25. $\int \sin\sqrt{x}\,dx$

Substitution: $y = \sqrt{x} \Rightarrow x = y^2 \Rightarrow dx = 2y\,dy$

$\int \sin\sqrt{x}\,dx = \int \sin y\cdot 2y\,dy = 2\int y\sin y\,dy$

Integration by parts:

$u = y \quad dv = \sin y\, dy$

$\Rightarrow du = dy \quad v = -\cos y$

so $\int y \sin y\, dy = -y\cos y - \int (-\cos y)\, dy = -y\cos y + \int \cos y\, dy$

$= -y\cos y + \sin y + C$

so $\int \sin\sqrt{x}\, dx = 2\int y \sin y\, dy = -2y\cos y + 2\sin y + C' = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C'$

37 want to prove $\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$

proof: $u = (\ln x)^n, \quad dv = dx$

$\Rightarrow du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx \quad v = x$

so $\int (\ln x)^n dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} \frac{1}{x} dx$

$= x(\ln x)^n - n\int (\ln x)^{n-1} dx$

[A Principle for Integration by Parts]

LIATE principle of precedence for u:

Logarithmic → Inverse trigonometric → Algebraic → Trigonometric → Exponential

If the integral has several factors, then we try to choose among them a "u" which appears as high as possible on the list. For example, in $\int xe^{2x} dx$ the integrand is xe^{2x}, which is the product of an algebraic function x and an exponential function e^{2x}. Since algebraic appears before exponential, we choose $u = x$. Sometimes the integration turns out to be similar regardless of the selection of u and dv, but it is advisable to refer to LIATE when in doubt.