Section Notes 11/29/07

P374 29 $f(x)=\frac{1}{x^2}$ is not continuous on the interval $[-1,3]$, so the Evaluation Theorem does not apply. In fact, f has an infinite discontinuity at $x=0$, so $\int_{-1}^{3}\frac{1}{x^2}dx$ does not exist.

Rmk Evaluation Theorem can only be applied when the function is continuous.

P375 47. If $w'(t)$ is the rate of change of weight in pounds per year, then $w(t)$ represents the weight in pounds of the child at age t. We know from the net change theorem that $\int_5^{10} w'(t)\,dt = w(10)-w(5)$, so the integral represents the increase in the child's weight (in pounds) between the age of 5 and 10.

49. Since $r(t)$ is the rate at which oil leaks, we can write $r(t)=-V'(t)$, where $V(t)$ is the volume of oil at time t. (Note that the minus sign is needed because V is decreasing, so $V'(t)$ is negative, but $r(t)$ is positive.) Thus, by the net change theorem, $\int_0^{120} r(t)\,dt = -\int_0^{120} V'(t)\,dt = -(V(120)-V(0))$ $=V(0)-V(120)$, which is the number of gallons of oil that leaked from the tank in the first two hours (120 minutes).

55 (a) Displacement = net change of position $= \int_0^3 v(t)\,dt$

$$= \int_0^3 (3t-5)\,dt = \left(\frac{3}{2}t^2-5t\right)\Big|_0^3 = \frac{27}{2}-15 = -\frac{3}{2}\text{ m}$$

(b) Distance Travelled $= \int_0^3 |v(t)|\,dt = \int_0^3 |3t-5|\,dt$

Note that $|3t-5| = \begin{cases} 5-3t & \text{when } t\le\frac{5}{3} \\ 3t-5 & \text{when } t\ge\frac{5}{3}\end{cases}$

so $\int_0^3 |3t-5|\,dt = \int_0^{\frac{5}{3}}(5-3t)\,dt + \int_{\frac{5}{3}}^3 (3t-5)\,dt$

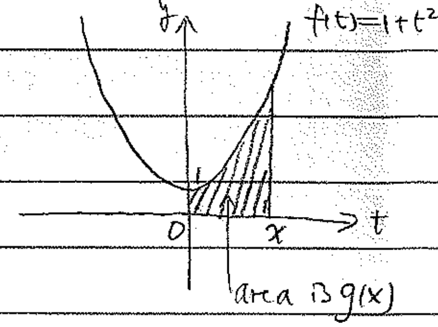
$$= \left(5t-\frac{3}{2}t^2\right)\Big|_0^{\frac{5}{3}} + \left(\frac{3}{2}t^2-5t\right)\Big|_{\frac{5}{3}}^3 = \dots = \frac{41}{6}\text{ m}$$

Recall Fundamental Theorem of Calculus

If f is continuous on $[a,b]$

(1) If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$

(2) $\int_a^b f(x)=F(b)-F(a)$, where F is any antiderivative of f, that is $F'=f$



P383 5. (a) By FTC1, $g(x)=\int_0^x (1+t^2)\,dt$

$$\Rightarrow g'(x)=f(x)=1+x^2$$

(b) By FTC2, $g(x)=\int_0^x (1+t^2)\,dt$

$$= t+\frac{1}{3}t^3\Big|_0^x$$

$$= (x+\frac{1}{3}x^3)-0 = x+\frac{1}{3}x^3$$

$$\Rightarrow g'(x)= 1+x^2$$

11. Let $u=\frac{1}{x}$ then $\frac{dh}{dx}=\frac{dh}{du}\frac{du}{dx}$

$h=\int_2^{\frac{1}{x}} \arctan t\,dt = \int_2^u \arctan t\,dt \Rightarrow \frac{dh}{du}=\arctan u=\arctan\frac{1}{x}$ (FTC1)

$u=\frac{1}{x} \Rightarrow \frac{du}{dx}=-\frac{1}{x^2}$ so $\frac{dh}{dx}=-\frac{1}{x^2}\arctan\frac{1}{x}$

15. $g(x)=\int_{2x}^{3x}\frac{u^2-1}{u^2+1}\,du=\int_{2x}^{0}\frac{u^2-1}{u^2+1}\,du+\int_0^{3x}\frac{u^2-1}{u^2+1}\,du=-\int_0^{2x}\frac{u^2-1}{u^2+1}\,du+\int_0^{3x}\frac{u^2-1}{u^2+1}\,du$

Let $g_1(x)=\int_0^{2x}\frac{u^2-1}{u^2+1}\,du$ and $g_2(x)=\int_0^{3x}\frac{u^2-1}{u^2+1}\,du$

then $g(x)=-g_1(x)+g_2(x)$ By FTC1,

$g_1'(x)=\frac{(2x)^2-1}{(2x)^2+1}\cdot\frac{d}{dx}(2x)=2\cdot\frac{4x^2-1}{4x^2+1}$ $g_2'(x)=\frac{(3x)^2-1}{(3x)^2+1}\cdot\frac{d}{dx}(3x)=3\cdot\frac{9x^2-1}{9x^2+1}$

so $g'(x)=-g_1'(x)+g_2'(x)=-2\frac{4x^2-1}{4x^2+1}+3\frac{9x^2-1}{9x^2+1}$

P384 26 (a) If $x<0$ then $g(x)=\int_0^x f(t)\,dt = \int_0^x 0\,dt = 0$

If $0 \le x \le 1$ then $g(x) = \int_0^x f(t)\,dt = \int_0^x t\,dt = \frac{1}{2}t^2\Big|_0^x = \frac{1}{2}x^2$

If $1<x\le 2$ then $g(x) = \int_0^x f(t)\,dt = \int_0^1 t\,dt + \int_1^x (2-t)\,dt$

$$= \frac{1}{2}t^2\Big|_0^1 + \left(2t-\frac{1}{2}t^2\right)\Big|_1^x = \frac{1}{2} + \left(2x-\frac{1}{2}x^2\right) - \left(2-\frac{1}{2}\right)$$

$$= 2x - \frac{1}{2}x^2 - 1$$

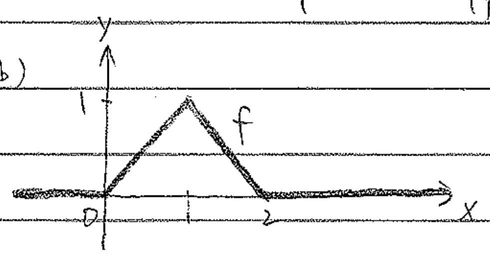
If $x>2$, then $g(x) = \int_0^x f(t)\,dt = \int_0^2 f(t)\,dt + \int_2^x f(t)\,dt$

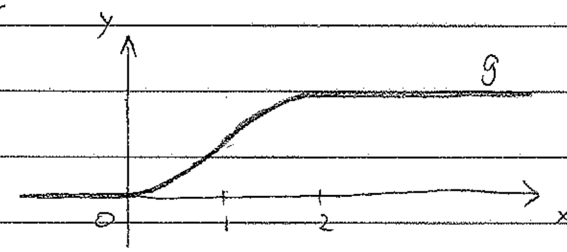
$$= g(2) + \int_2^x 0\,dt = \left(2\times 2 - \frac{1}{2}2^2 - 1\right) + 0 = 1$$

so

$$g(x) = \begin{cases} 0 & \text{if } x<0 \\ \frac{1}{2}x^2 & \text{if } 0\le x\le 1 \\ 2x-\frac{1}{2}x^2-1 & \text{if } 1<x\le 2 \\ 1 & \text{if } x>2 \end{cases}$$

(b)





(c) f is not differentiable at its corners at $x=0, 1$ and 2

So f is differentiable on $(-\infty,0)\cup(0,1)\cup(1,2)\cup(2,\infty)$

g is differentiable on $(-\infty,\infty)$. (Actually by FTC I, $g'(x)=f(x)$.)

27. Using FTC I, we differentiate both sides of $6+\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x}$ w.r.t x.

We get: $\frac{f(x)}{x^2} = 2\cdot\frac{1}{2\sqrt{x}} \Rightarrow f(x) = x^{\frac{3}{2}}$

To find a, we substitute $x=a$ in the original equation to obtain

$6+\int_a^a \frac{f(t)}{t^2}\,dt = 2\sqrt{a} \Rightarrow 6+0 = 2\sqrt{a} \Rightarrow 3=\sqrt{a} \Rightarrow a=9$