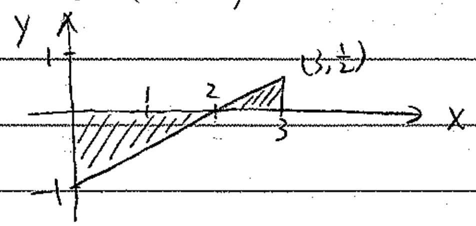
Section Notes 11/27/07

[1] Definite integral = net area between the graph and x-axis

Example: P366, 33

Evaluate $\int_0^3 \left(\frac{1}{2}x-1\right) dx$

Graph:



Net area = area of the small triangle − area of the large triangle

$= \frac{1}{2}\cdot 1\cdot \frac{1}{2} - \frac{1}{2}\cdot 2\cdot 1 = \frac{1}{4} - 1 = -\frac{3}{4}$

[2] Definite integral = limit of Riemann Sum

Recall: $\int_a^b f(x)\,dx$ definite integral

The interval $[a,b]$ can be divided averagely into n parts.

the seperating points: $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

length of each small interval: $\Delta x = \frac{b-a}{n}$

use left endpoints, Riemann sum $L_n = \sum_{i=0}^{n-1} f(x_i)\cdot\Delta x = \sum_{i=1}^{n} f(x_{i-1})\cdot\Delta x$

use right endpoints, Riemann sum $R_n = \sum_{i=1}^{n} f(x_i)\cdot\Delta x$

and $\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ where x_i^* are sample points

$\boxed{\text{Type I problems}}$:

Realize the limit of a Riemann sum as a definite integral

Example: P353. 19

$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{4n}\tan\frac{i\pi}{4n}$

The two factors "$\frac{\pi}{4n}$" and "$\tan\frac{i\pi}{4n}$" should be $f(x_i^*)$ and Δx

Note that Δx is independent of i, so $\Delta x = \frac{\pi}{4n}$

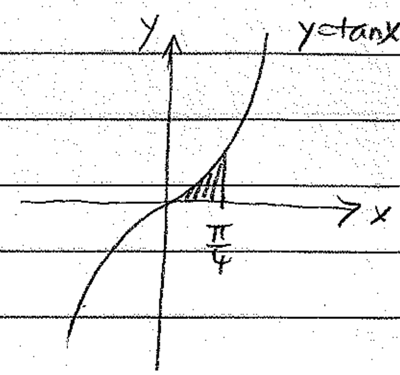
and $f(x_i^*) = \tan\frac{i\pi}{4n} \Rightarrow x_i^* = \frac{i\pi}{4n}$, $f(x) = \tan x$

Note $x_i^* = \frac{i\pi}{4n}$ are right endpoints, so

$x_0 = 0,\ x_1 = \frac{\pi}{4n},\ \cdots,\ x_n = \frac{n\pi}{4n}$

$a = x_0 = 0,\ b = x_n = \frac{\pi}{4}$. so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n} = \int_0^{\frac{\pi}{4}} \tan x\, dx$$



Example P365 17.

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i \sin x_i\, \Delta x \quad \text{on } [0,\pi]$$

Obviously, $f(x_i^*) = x_i \sin x_i$ $\quad x_i^* = x_i$ right endpoints, $f(x) = x\sin x$

so $\lim_{n\to\infty} \sum_{i=1}^{n} x_i \sin x_i\, \Delta x_i = \int_0^{\pi} x \sin x\, dx$

$\boxed{\text{Type II problems}}$:

Evaluate a definite integral via Riemann Sums.

Example: P365. 23

$$\int_0^2 (2-x^2)\,dx$$

Interval $[0,2]$ n small intervals, $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

so seperating points: $x_0 = 0,\ x_1 = \frac{2}{n},\ x_2 = \frac{4}{n},\ \cdots,\ x_n = 2$ or say $x_i = \frac{2i}{n}$

function $f(x) = 2 - x^2$

Use right endpoints: $R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \left(2 - \left(\frac{2i}{n}\right)^2\right)\cdot\left(\frac{2}{n}\right)$

$= \sum_{i=1}^{n} \left(\frac{4}{n} - \frac{8i^2}{n^3}\right) = \frac{4}{n}\cdot n - \frac{8}{n^3}\sum_{i=1}^{n} i^2 = 4 - \frac{8}{n^3}\cdot\frac{1}{6}n(n+1)(2n+1) = 4 - \frac{4}{3}\frac{(n+1)(2n+1)}{n^2}$

So $\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(4 - \frac{4}{3}\frac{(n+1)(2n+1)}{n^2}\right) = 4 - \frac{4}{3}\cdot 2 = \frac{4}{3}$

$\boxed{3}$ Properties of definite integrals

- linearity $\quad \int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha\int_a^b f(x)\,dx + \beta\int_a^b g(x)\,dx$
- adjacent intervals $\quad \int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- comparison $\quad$ If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$
- Switching limits $\quad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

[4] Evaluation theorem:

If $F(x)$ is an antiderivative of $f(x)$, and $f(x)$ is continuous on $[a,b]$, then $\int_a^b f(x)dx = F(b)-F(a)$

Examples: P374 15, 21

15 $\int_0^{\frac{\pi}{4}} \sec^2 t\, dt = \tan t \Big|_0^{\frac{\pi}{4}} = \tan\frac{\pi}{4} - \tan 0 = 1-0=1$

21. $\int_1^{64} \frac{1+\sqrt[3]{x}}{\sqrt{x}}dx = \int_1^{64}\left(\frac{1}{\sqrt{x}} + \frac{\sqrt[3]{x}}{\sqrt{x}}\right)dx = \int_1^{64}\left(x^{-\frac{1}{2}} + x^{-\frac{1}{6}}\right)dx$

$= \left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + \frac{x^{-\frac{1}{6}+1}}{-\frac{1}{6}+1}\right)\Bigg|_1^{64} = \left(2x^{\frac{1}{2}} + \frac{6}{5}x^{\frac{5}{6}}\right)\Bigg|_1^{64}$

$= \left(2\cdot 64^{\frac{1}{2}} + \frac{6}{5}64^{\frac{5}{6}}\right) - \left(2\cdot 1^{\frac{1}{2}} + \frac{6}{5}1^{\frac{5}{6}}\right) = \left(16+\frac{6}{5}\cdot 32\right) - \left(2+\frac{6}{5}\right) = \frac{256}{5}$

[5] Indefinite Integrals revisited.

Example: P374 37

verify $\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1}+C$ by differentiation

The problem says: $\sqrt{x^2+1}$ is an antiderivative of $\frac{x}{\sqrt{x^2+1}}$

So just need to verify: $\frac{x}{\sqrt{x^2+1}}$ is the derivative of $\sqrt{x^2+1}$

Check: $\frac{d}{dx}\left(\sqrt{x^2+1}\right) = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot(2x) = \frac{x}{\sqrt{x^2+1}}$, as desired