Section Notes 11/08/07

1 Newton's Method

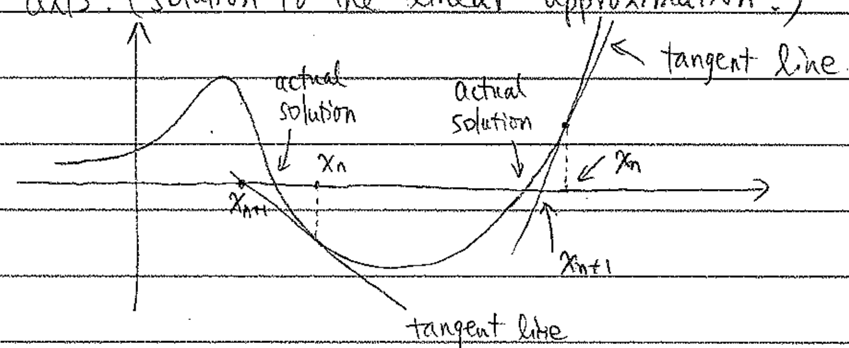
Pay attention to 3 points:

① It's a numerical method for finding solutions to equations $f(x)=0$

② The initial approximation should be "close enough" to the actual solution.

③ At each step, the new approximation of the solution to the original equation is the intersection of the tangent line and x axis. (solution to the linear approximation.)

Picture



Formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Examples P325–327

3. x_2 = solution of the equation of the tangent line at $(x_1, f(x_1))$

$y = 5x-4$

solve $5x-4=0$ for $x \Rightarrow x = \frac{4}{5}$ so $x_2 = \frac{4}{5}$

For 9, 11, 27

Step 1. Which equation do we want to solve?

Step 2. Can you guess an initial approximation to the solution?

Step 3. Run Newton's method to compute the solution (till the requirement is satisfied)

(Remark: the better your initial value is, the sooner you can get the required accuracy.)

9 Step 1. We want to solve $x^3-30=0$ for x.

Step 2. It seems that the solution is slightly larger than 3. So $x_1=3$ might be a good choice. Or $x_1=3.1$ is also a good choice.

Step 3. Find the iteration formula

$$f(x)=x^3-30 \qquad f'(x)=3x^2$$

$$\text{so } x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{x_n^3-30}{3x_n^2}$$

Let's start from $x_1=3$ and we need to find approximations until they agree to eight decimal places.

$x_1=3$

$\Rightarrow x_2 \approx 3.11111111$

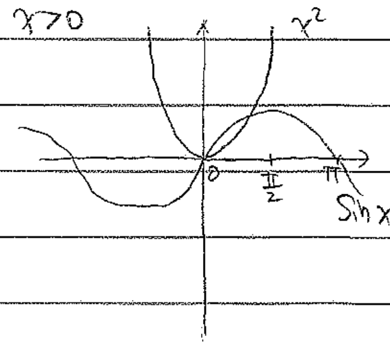
$\Rightarrow x_3 \approx 3.10723734$

$\Rightarrow x_4 \approx 3.10723251$

$\Rightarrow x_5 \approx 3.10723251$

so $\sqrt[3]{30} \approx 3.10723251$, correct to eight decimal places.

11 Step 1 We want to solve $\sin x - x^2=0$ for $x>0$



Step 2. Guess where the solution is, by graph.

It seems that the root is between 0 and $\frac{\pi}{2}$, approximately equal to 1. So set $x_1=1$

Step 3. Iteration formula: $f(x)=\sin x - x^2$

$$\Rightarrow f'(x)=\cos x-2x$$

$$\text{so } x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\sin x_n - x_n^2}{\cos x_n-2x_n}$$

Please finish off the computation yourself.

27 Step 1. want to look for inflection point:

x coordinate of inflection pt is the solution to $y''=0$.

$y=e^{\cos x} \Rightarrow y'=e^{\cos x}\cdot(-\sin x) \Rightarrow y''=e^{\cos x}\sin^2 x-e^{\cos x}\cos x$

So, want to solve $e^{\cos x}\sin^2 x-e^{\cos x}\cos x=0$ for x in $[0,\pi]$

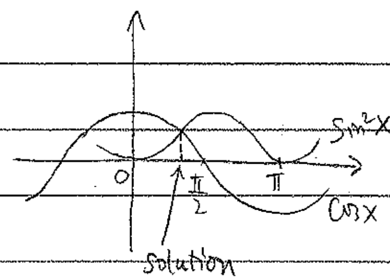
Notice that the function $e^{\cos x}\sin^2 x-e^{\cos x}\cos x=e^{\cos x}(\sin^2 x-\cos x)$

which $e^{\cos x}$ is always positive, the solution must satisfy

$\sin^2 x-\cos x=0$. Use this as our new equation to make computation simpler.

Step 2. Guess the initial approximation of the solution to $\sin^2 x-\cos x=0$ by graph.

It seems that the solution is very close to 1, so set $x_1=1$



Step 3. Iteration formula:

$f(x)=\sin^2 x-\cos x \Rightarrow f'(x)=2\sin x\cos x+\sin x$

So $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{\sin^2 x_n-\cos x_n}{2\sin x_n\cos x_n+\sin x_n}$

$x_1=1$

$\Rightarrow x_2\approx 0.904173$

$\Rightarrow x_3\approx 0.904557$

$\Rightarrow x_4\approx 0.904557$

So the solution (x-coordinates of inflection point) is approximately 0.904557. From $y=e^{\cos x}$ we have the y-coordinate of inflection point is $e^{\cos 0.904557}\approx 1.855277$

So the inflection pt is approximately at $(0.904557, 1.855277)$

[2] L'Hospital's rule.

Things you need to know:

① L'Hospital's rule can only be used to indeterminate forms.

i.e. indeterminate quotients: $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

indeterminate products: $0\cdot(\pm\infty)$ or $(\pm\infty)\cdot 0$

indeterminate difference: $(+\infty)-(+\infty)$ or $(-\infty)-(-\infty)$

indeterminate powers: 0^0, $(\pm\infty)^0$, or 1^∞

② method to use in each situation:

indeterminate quotient $\Rightarrow$ L'Hospital

indeterminate product $f\cdot g \Rightarrow \frac{f}{\frac{1}{g}}$ or $\frac{g}{\frac{1}{f}} \Rightarrow$ L'Hospital.

indeterminate difference $f-g \Rightarrow$ factor into product (or quotient)

$\Rightarrow$ follow the way for product (or quotient)

indeterminate powers $f^g \Rightarrow$ take natural log to get

$\ln(f^g) = g\ln f \Rightarrow$ follow the way for product to find the limit of $g\ln f \Rightarrow$ take exponential to get the original limit

Route Map: $\boxed{\text{Ind. Difference}} \rightarrow \boxed{\text{Ind Product}} \rightarrow \boxed{\text{Ind Quotient}} \rightarrow$ L'Hospital

$\boxed{\text{Ind. Power}} \rightarrow \boxed{\text{Ind Product}}$

Examples P303–304.

13. when $x\to 0^+$, $\ln x \to -\infty$, $x\to 0^+$ not indeterminate form.

(this form is $\frac{-\infty}{0^+}$) the limit is $-\infty$ because dividing by small values of x just increase the magnitude of the quotient.

L'Hospital's rule doesn't apply

17 when $x \to 0$ $\quad e^x - 1 - x \to 1 - 1 - 0 = 0$. $\quad x^2 \to 0$

$\Rightarrow$ Indeterminate Quotient $\frac{0}{0} \Rightarrow$ L'Hospital's rule

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x\to 0} \frac{(e^x - 1 - x)'}{(x^2)'} = \lim_{x\to 0} \frac{e^x - 0 - 1}{2x} = \lim_{x\to 0} \frac{e^x - 1}{2x}$$

when $x \to 0$, $\quad e^x - 1 \to 1 - 1 = 0 \quad 2x \to 0$

$\Rightarrow$ Indeterminate Quotient $\Rightarrow$ L'Hospital's rule again

$$\lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{(e^x - 1)'}{(2x)'} = \lim_{x\to 0} \frac{e^x - 0}{2} = \lim_{x\to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$$

(Continuity of $\frac{e^x}{2}$)

19 When $x \to \infty$, $x \to \infty$,

$1 + 2e^x \to \infty \Rightarrow \ln(1 + 2e^x) \to \infty$

$\Rightarrow$ Indeterminate form $\frac{\infty}{\infty} \Rightarrow$ L'Hospital.

$$\lim_{x\to\infty} \frac{x}{\ln(1+2e^x)} = \lim_{x\to\infty} \frac{1}{\frac{1}{1+2e^x}\cdot(0+2e^x)} = \lim_{x\to\infty} \frac{1+2e^x}{2e^x}$$

Use L'Hospital's rule or not to compute this limit. If use,

$2e^x \to \infty$, $\quad 1 + 2e^x \to \infty \quad \Rightarrow$ indeterminate quotient $\frac{\infty}{\infty} \Rightarrow$ L'Hospital.

$$\lim_{x\to\infty} \frac{1+2e^x}{2e^x} = \lim_{x\to\infty} \frac{(1+2e^x)'}{(2e^x)'} = \lim_{x\to\infty} \frac{0+2e^x}{2e^x} = \lim_{x\to\infty} 1 = 1$$

31 When $x \to \infty$, $\quad e^{\frac{1}{x}} \to e^0 = 1$ so $x e^{\frac{1}{x}} \to \infty$, $\quad x \to \infty$

$\Rightarrow$ Indeterminate difference $\infty - \infty$

Factor: $\quad x e^{\frac{1}{x}} - x = x(e^{\frac{1}{x}} - 1)$

$x \to \infty$ and $e^{\frac{1}{x}} \to 1$ so $e^{\frac{1}{x}} - 1 \to 0$

$\Rightarrow$ Indeterminate quotient

Write it as a quotient. $\quad x(e^{\frac{1}{x}} - 1) = \dfrac{e^{\frac{1}{x}} - 1}{\frac{1}{x}}$

now $\frac{1}{x} \to 0$, $\quad e^{\frac{1}{x}} - 1 \to 0$

$\Rightarrow$ Indeterminate quotient $\Rightarrow$ L'Hospital's rule.

$$\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)=\lim_{x\to\infty} x\left(e^{\frac{1}{x}}-1\right)=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\left(e^{\frac{1}{x}}-1\right)'}{\left(\frac{1}{x}\right)'}$$

$$=\lim_{x\to\infty}\frac{e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}=\lim_{x\to\infty}e^{\frac{1}{x}}=\lim_{t\to 0}e^{t}=e^{0}=1$$

(continuity of e^t)

33. when $x\to\infty$, $\ln x\to\infty$ $\Rightarrow$ indeterminate form $\infty-\infty$.

factor: $x-\ln x = x\left(1-\frac{\ln x}{x}\right)$

Is it an indeterminate product? $\Rightarrow$ need to evaluate the limit of $\frac{\ln x}{x}$

$\Rightarrow$ but $\frac{\ln x}{x}$ itself is an indeterminate form $\frac{\infty}{\infty}$ $\Rightarrow$ L'Hospital.

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{(\ln x)'}{x'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=\lim_{x\to\infty}\frac{1}{x}=0$$

so $\lim_{x\to\infty}\left(1-\frac{\ln x}{x}\right)=1-0=1$

so $x\left(1-\frac{\ln x}{x}\right)$ is of the form $\infty\cdot 1$, not an indeterminate product

(cannot apply L'Hospital) but $\lim_{x\to\infty} x\left(1-\frac{\ln x}{x}\right)=\infty$

so $\lim_{x\to\infty}(x-\ln x)=\infty$

37. when $x\to 0$, $1-2x\to 1$ $\quad \frac{1}{x}\to\infty$ $\quad$ form 1^{∞} $\Rightarrow$ indeterminate

$\Rightarrow$ take natural log

$$y=(1-2x)^{\frac{1}{x}} \Rightarrow \ln y=\frac{1}{x}\ln(1-2x)=\frac{\ln(1-2x)}{x}$$

since $x\to 0$ $\Rightarrow$ $1-2x\to 1$ $\Rightarrow$ $\ln(1-2x)\to 0$, $\frac{0}{0}$ indeterminate form

$\Rightarrow$ L'Hospital's rule

$$\lim_{x\to 0}\ln y=\lim_{x\to 0}\frac{\ln(1-2x)}{x}=\lim_{x\to 0}\frac{(\ln(1-2x))'}{x'}=\lim_{x\to 0}\frac{\frac{1}{1-2x}\cdot(-2)}{1}$$

$$=\lim_{x\to 0}\frac{-2}{1-2x}=\frac{-2}{1-2\cdot 0}=-2$$

continuity of $\frac{-2}{1-2x}$ at $x=0$

so $\lim_{x\to 0}(1-2x)^{\frac{1}{x}}=\lim_{x\to 0}y=\lim_{x\to 0}e^{\ln y}=e^{\lim_{x\to 0}\ln y}=e^{-2}$