Section Notes 11/06/07

Optimizations:

Step 1 Write down the function (that you want to extremize) and all relations of variables in the form of equations.

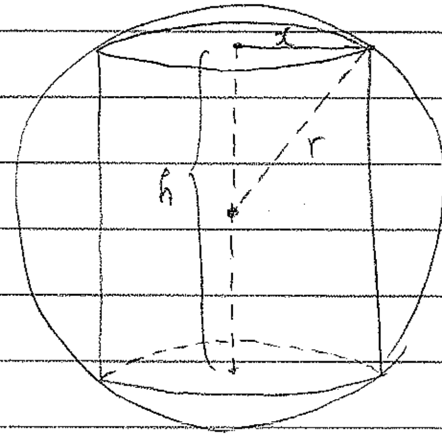
Step 2 Use the relations to eliminate all redundant variables to obtain a single-variable function

Step 3 Find the domain of the function

Step 4 Find the max/min of the function on the given domain

Suggestion: if the domain is a closed interval, use closed interval method; if an open interval, use first derivative test (or second derivative test)

Examples P311 ~ 314 17, 45, 21, 19



17. Step 1.

Goal: Maximize $V = \pi x^2 \cdot h$

Relation: $r^2 = x^2 + \left(\frac{h}{2}\right)^2$

Step 2.

$x^2 = r^2 - \left(\frac{h}{2}\right)^2$ eliminate x

$V(h) = \pi \cdot \left(r^2 - \frac{h^2}{4}\right) \cdot h = \pi r^2 h - \frac{\pi}{4} h^3$

(where r is a constant)

Step 3 domain: $0 < h < 2r$

Step 4 open interval $\Rightarrow$ first derivative test

$V'(h) = \pi r^2 - \frac{3\pi}{4} h^2$

$V'(h) = 0 \Rightarrow \pi r^2 = \frac{3\pi}{4} h^2 \Rightarrow h^2 = \frac{4}{3} r^2 \Rightarrow h = \pm \frac{2}{\sqrt{3}} r$

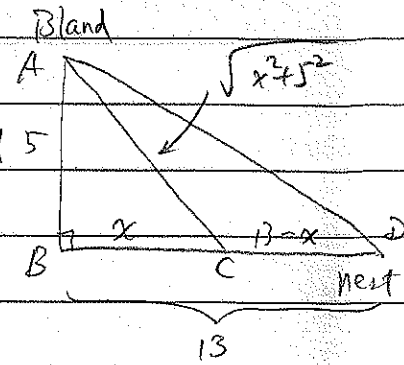
the only critical point in $(0, 2r)$ is $h = \frac{2}{\sqrt{3}} r$.

when $0 < h < \frac{2}{\sqrt{3}} r$, $V'(h) = \pi r^2 - \frac{3\pi}{4} h^2 > \pi r^2 - \frac{3\pi}{4} \left(\frac{2}{\sqrt{3}} r\right)^2 = 0$

when $\frac{2}{\sqrt{3}} r < h < 2r$, $V'(h) = \pi r^2 - \frac{3\pi}{4} h^2 < \pi r^2 - \frac{3\pi}{4} \left(\frac{2}{\sqrt{3}} r\right)^2 = 0$

By first derivative test, $V\left(\frac{2}{\sqrt{3}} r\right)$ is the maximum

$$V\left(\frac{2}{\sqrt{3}}r\right) = \pi\left(r^2 - \frac{1}{4}\left(\frac{2}{\sqrt{3}}r\right)^2\right)\cdot\frac{2}{\sqrt{3}}r = \frac{\pi}{3\sqrt{3}}r^3$$



45(a) Goal: Maximize total Energy

E = energy spent on water + energy spent on land

route AC is on water.

route CD is on land.

assume: energy spent per km on land: k

then: energy spent per km on water: $1.4k$

$E = 1.4k \cdot \sqrt{x^2+5^2} + k\cdot(13-x)$

Domain: $0 \le x \le 13$

Maximize E on the closed interval $\Rightarrow$ close interval method.

$E'(x) = 1.4k \dfrac{x}{\sqrt{x^2+5^2}} - k$

critical pts: $E'(x)=0 \Rightarrow 1.4k\dfrac{x}{\sqrt{x^2+5^2}} = k \Rightarrow 1.4x = \sqrt{x^2+5^2}$

$\Rightarrow (1.4x)^2 = x^2+25 \Rightarrow 0.96x^2 = 25 \Rightarrow x = \pm\sqrt{\frac{25}{0.96}}$

But the only critical point in $[0,13]$ is $\sqrt{\frac{25}{0.96}} \approx 5.1$

Closed interval method:

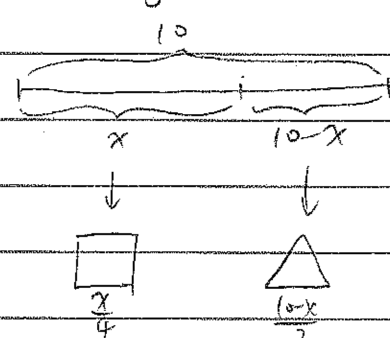
$E(0) = 1.4k\sqrt{0^2+5^2} + 13k = 20k$

$E\left(\sqrt{\frac{25}{0.96}}\right) = \dots \approx 17.9k$

$E(13) = 1.4k\sqrt{13^2+5^2} + 0 \approx 19.5k$

So $E\left(\sqrt{\frac{25}{0.96}}\right) \approx 17.9k$ is the smallest energy.

So the point C should be approximately 5.1 km away from B.



21. Goal: maximize/minimize total area

A = area of square + area of triangle

area of square $= \left(\frac{x}{4}\right)^2$

area of triangle $= \frac{1}{2}\cdot\frac{10-x}{3}\cdot\underbrace{\frac{10-x}{6}\sqrt{3}}$

height can be computed from Pythagorean theorem.

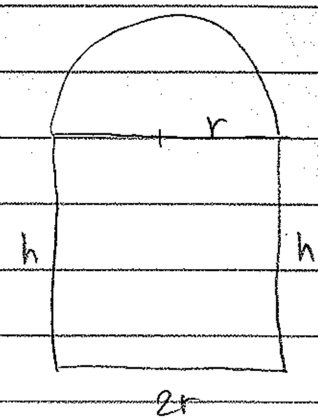
So $A=\left(\frac{x}{4}\right)^2+\frac{1}{2}\frac{10-x}{3}\cdot\frac{10-x}{6}\sqrt{3}$

Domain: $0\le x\le 10$

closed interval $\Rightarrow$ use closed interval method.

Finish the computation yourself.

19. Goal: maximize area



$A=\frac{1}{2}\pi r^2+2r\cdot h$

relation: perimeter $=30$, i.e.

$\pi r+2h+2r=30$

eliminate h: $h=\frac{30-(\pi+2)r}{2}$

$A=\frac{1}{2}\pi r^2+2r\cdot\frac{30-(\pi+2)r}{2}$

$=\frac{1}{2}\pi r^2+r(30-(\pi+2)r)$

Domain: first of all. $r>0$

second of all, $h>0$, so $\frac{30-(\pi+2)r}{2}>0\Rightarrow r<\frac{30}{\pi+2}$

so domain is $(0,\frac{30}{\pi+2})$

Critical pts: $A'(r)=\pi r+30-2(\pi+2)r=30-(\pi+4)r$

$A'(r)=0\Rightarrow r=\frac{30}{\pi+4}$ the only critical point in $(0,\frac{30}{\pi+2})$

You can use first or second derivative test.

If use the second derivative test, compute $A''(r)$

$A''(r)=-(\pi+4)<0$.

so $A(\frac{30}{\pi+4})$ is the maximum of the function

The problem wants us to find the dimension of the window.

that is: radius of semicircle $=r=\frac{30}{\pi+4}$ ft,

base of rectangle $=2r=\frac{60}{\pi+4}$ ft,

height of rectangle $=\frac{1}{2}(30-(\pi+2)r)=\cdots=\frac{30}{\pi+4}$ ft.