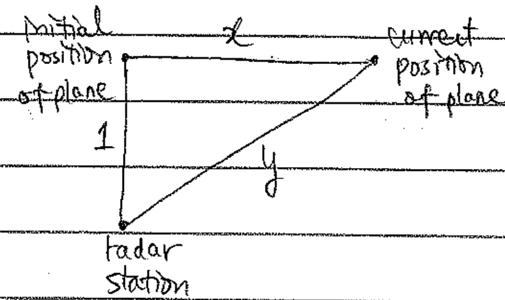
Section Notes 10/30/07

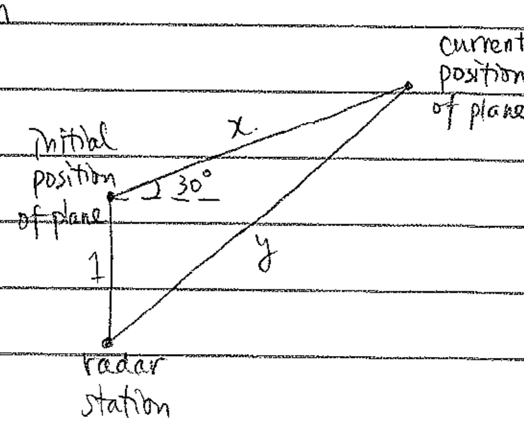
I spreaded two handouts in sections today.
One handout gives the complete solutions to the related rates problems 9–12
The other explains the "error" issues with a couple of examples.
Both of the handouts are downloadable from my webpage, as usual.

What heppened in sections today: related rates.
problems 11, 35, 21, 10, 12



11 Functions: x: position function of plane.
y: distance between radar and plane.
Given. $\frac{dx}{dt} = 500$
Unknown: $\frac{dy}{dt} = ?$ when $y = 2$

Equation: $y^2 = x^2 + 1^2$
Differentiate both sides with respect to t
$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} \quad \text{so} \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$
We have $y = 2$ and $\frac{dx}{dt} = 500$
From the equation $y^2 = x^2 + 1^2$ together with $y = 2$ we can solve $x = \sqrt{3}$
So $\frac{dy}{dt} = \frac{\sqrt{3}}{2} \cdot 500 = 250\sqrt{3}$ mi/h ≈ 433 mi/h



35 Functions: x: position function of plane
y: distance between radar and plane
Given: $\frac{dx}{dt} = 300$ km/h
Unknown. $\frac{dy}{dt} = ?$ when $t = 1$ min $= \frac{1}{60}$ h

Equation: By law of cosine,

$$y^2 = x^2 + 1^2 - 2\cdot x\cdot 1\cdot\cos 120^\circ$$
$$= x^2 + 1^2 - 2\cdot x\cdot 1\cdot\left(-\tfrac{1}{2}\right)$$
$$= x^2 + 1 + x.$$

Differentiate both sides with respect to t,

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt} + \frac{dx}{dt}$$
$$= (2x+1)\frac{dx}{dt} \qquad \text{so} \quad \frac{dy}{dt} = \frac{2x+1}{2y}\frac{dx}{dt}$$

We have $\frac{dx}{dt} = 300$,

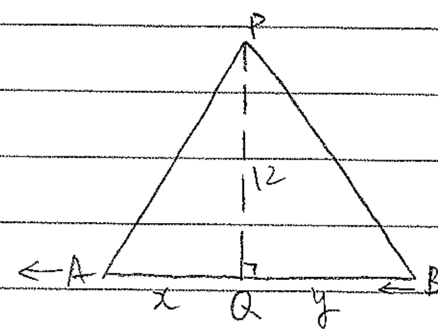
From $t = \frac{1}{60}$ we can compute $x = 300\times\frac{1}{60} = 5$

From the equation $y^2 = x^2 + 1 + x$ together with $x = 5$ we can solve $y = \sqrt{31}$

So $\frac{dy}{dt} = \frac{2\times 5+1}{2\cdot\sqrt{31}}\cdot 300 = \frac{1650}{\sqrt{31}} \approx 296$ km/h

21. Functions: x: length of AQ

y: length of BQ



Given: $PA + PB = 39$

$\frac{dx}{dt} = 2$

Unknown: $\frac{dy}{dt} = ?$ when $x = 5$

Caution: Because both A and B are moving toward left, y is decreasing. We should expect $\frac{dy}{dt}$ be negative. However, we have

Speed of cart B = rate at which y decreases $= -\frac{dy}{dt}$

Equation: In the right angled triangle QPA, we have

$$PA^2 = x^2 + 12^2 \Rightarrow PA = \sqrt{x^2+12^2}$$

In the right angled triangle QPB, we have

$$PB^2 = y^2 + 12^2 \Rightarrow PB = \sqrt{y^2+12^2}$$

We have the condition $PA + PB = 39$, therefore

$$\sqrt{x^2+12^2} + \sqrt{y^2+12^2} = 39$$

Differentiate both sides with respect to t.

$$\frac{1}{2}(x^2+12^2)^{-\frac{1}{2}} \cdot 2x \cdot \frac{dx}{dt} + \frac{1}{2}(y^2+12^2)^{-\frac{1}{2}} \cdot 2y \cdot \frac{dy}{dt} = 0$$

i.e. $$\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt} + \frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{\sqrt{y^2+12^2}}{y} \cdot \frac{x}{\sqrt{x^2+12^2}} \cdot \frac{dx}{dt}$$

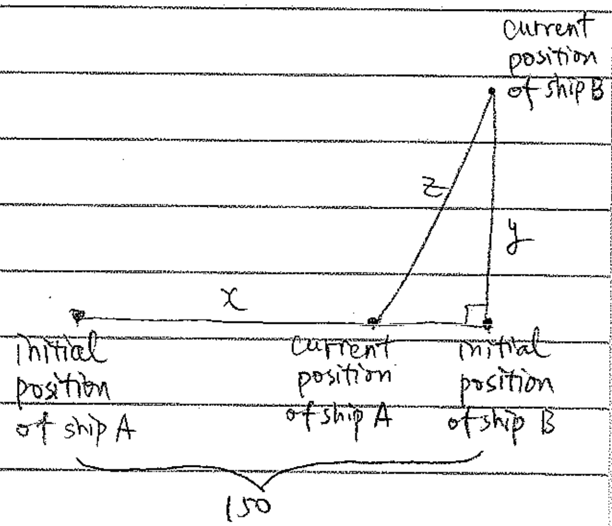
We have $\frac{dx}{dt} = 2$ and $x = 5$

From the equation $\sqrt{x^2+12^2} + \sqrt{y^2+12^2} = 39$ together with $x=5$

we can get $y = \sqrt{532}$.

So $$\frac{dy}{dt} = -\frac{\sqrt{532+144}}{\sqrt{532}} \cdot \frac{5}{\sqrt{25+144}} \cdot 2 = -\frac{10}{\sqrt{133}} \text{ ft/s} \approx -0.87 \text{ ft/s}$$

As we said in the beginning, the speed of cart B is the opposite of -0.87 ft/s, so the cart B is moving towards Q at about 0.87 ft/s.



10 Function: x: position function of ship A

y: position function of ship B

z: distance between two ships.

Given: $\frac{dx}{dt} = 35$

$\frac{dy}{dt} = 25$

Unknown $\frac{dz}{dt} = ?$ when $t = 4$

Equation: $z^2 = (150-x)^2 + y^2$

Differentiate both sides with respect to t:

$$2z\frac{dz}{dt} = 2(150-x)\cdot(-1)\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\Rightarrow \frac{dz}{dt} = -\frac{150-x}{z}\frac{dx}{dt} + \frac{y}{z}\frac{dy}{dt}$$

We have $\frac{dx}{dt} = 35$, $\frac{dy}{dt} = 25$.

When $t=4$, x = distance that ship A travelled in 4 hours $= 35 \times 4 = 140$

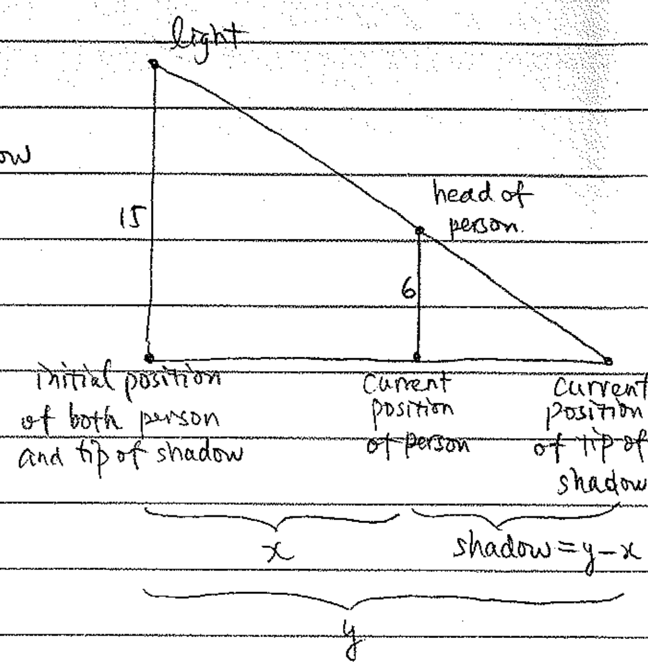
y = distance that ship B travelled in 4 hours $= 25 \times 4 = 100$

From the equation $z^2 = (150-x)^2 + y^2$ together with

$x = 140$ and $y = 100$ we can find value of z.

Then we are ready to compute $\frac{dz}{dt}$

Finish off the computation yourself



12. Function: x: position function of person

y: position function of tip of shadow

Given: $\frac{dx}{dt} = 5$

Unknown: $\frac{dy}{dt} = ?$ when $x = 40$

Equation:

From similar triangles, we have

$$\frac{15}{6} = \frac{y}{y-x}$$

So $15(y-x) = 6y$

$15y - 15x = 6y$

$9y = 15x$

Differentiate both sides with respect to t.

$$9\frac{dy}{dt} = 15\frac{dx}{dt} \quad \text{So} \quad \frac{dy}{dt} = \frac{15}{9}\frac{dx}{dt}$$

Plug in $\frac{dx}{dt} = 5$ and we get $\frac{dy}{dt} = \frac{15}{9} \cdot 5 = \frac{25}{3}$ ft/s

Note that $x = 40$ ft is irrelavant here.