Section Notes 10/25/07

1. Mean value theorem.

State the theorem, emphasis "f is differentiable on the closed interval $[a,b]$"

P288 53. (Pay attention to the way of writing down the solution)
Consider the interval $[0,4]$, on which $f(x)$ is differentiable.
By Mean Value Theorem, there exists some c between 0 and 4, such that $f'(c) = \frac{f(4)-f(0)}{4-0}$, i.e. $f(4) = f(0) + f'(c)\cdot(4-0)$
$= -3 + 4f'(c)$.
But since $f' \le 5$ always, we know in particular that no matter what value c is, $f'(c) \le 5$, so
$f(4) = -3 + 4f'(c) \le -3 + 4\cdot 5 = 17$
So the largest possible value for $f(4)$ is 17.

55. Let $g(t)$ and $h(t)$ be the position function of the two runners and let $f(t) = g(t) - h(t)$. Let $t=0$ be the starting time and $t=b$ be the finishing time, then $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$. Since both of $g(t)$ and $h(t)$ are differentiable in the interval $[0,b]$ (because at any time the derivative is the speed at that time.), $f(t) = g(t) - h(t)$ is differentiable as well and $f'(t) = g'(t) - h'(t)$.
By mean value theorem, there exists a time c between 0 and b, such that $f'(c) = \frac{f(b)-f(0)}{b-0} = \frac{0-0}{b-0} = 0$. So $f'(c) = g'(c) - h'(c) = 0$, and we have $g'(c) = h'(c)$. So at time c, both runners have the same speed $g'(c) = h'(c)$.

[2] linear approximation

Idea: use the function for the tangent line to approximate the original function. From the graphs we can see that they are indeed very close, so it is a good approximation.

Formula: $L(x) = f(a) + f'(a)\cdot(x-a)$ (when x is closed to a)

Advantage: ① good approximation

② need very little information (the values of the function and the derivative only)

Examples:

P252 7. $a = \frac{\pi}{2}$ $f(x) = \cos x \Rightarrow f(a) = f(\frac{\pi}{2}) = 0$

$f'(x) = -\sin x \Rightarrow f'(a) = f'(\frac{\pi}{2}) = -1$

$L(x) = f(a) + f'(a)\cdot(x-a) = 0 + (-1)\cdot(x - \frac{\pi}{2}) = -x + \frac{\pi}{2}$

15. $(2.001)^5$ is very hard to compute (without a calculator)

how to find an approximation for that?

Rephrase it as the following problem:

want to find the value of the function $f(x) = x^5$ at $x = 2.001$, but $f(2)$ is very easy to compute by hand, so try to use linear approximation at $x=2$ to compute $f(2.001)$

$f(2.001) \approx L(2.001) = \cdots$

($\uparrow$ cannot use "=")

33. Two points to state:

(1) In order to use the linear approximation, we don't need to know what the function is. All we need is the values of the function and its derivative at a point

For example, in this problem, $f(1) = 5$, $f'(1) = 2$, so

$f(x) \approx L(x) = f(1) + f'(1)(x-1) = 5 + 2(x-1)$

Therefore, $f(0.9) \approx L(0.9) = 5 + 2\cdot(0.9-1) = 4.8$

$f(1.1) \approx L(1.1) = 5+2\cdot(1.1-1) = 5.2$

(2) underestimate $\Leftrightarrow$ tangent line is below the graph of the function

$\Leftrightarrow$ function is concave upward

overestimate $\Leftrightarrow$ tangent line is above the graph of the function

$\Leftrightarrow$ function is concave downward.

(draw pictures)

For example, in this problem, f' decreasing $\Rightarrow$ f concave downward

so the linear approximation is overestimate

$\boxed{3}$ Error, Relative error,

(1) Error: function $y = f(x)$, at a certain point $\underbrace{}_{x=a}$ get the value of the function $(y = f(a))$

but, sometimes x cannot be measured accurately. The error in the independent variable x is the difference of the actual value of x and the measured value of x, called Δx, also called dx

If x is not accurate, y will not be accurate. This results in a difference of the actual value of y and the computed valued of y based on the "wrong" value of x.

So: actual value of y: $f(x+\Delta x)$

computed value of y: $f(x)$

difference: $f(x+\Delta x) - f(x)$ called the error in y, denoted by Δy

(2) Maximum Error. The larger the error in x is, the larger the error in y is.

Typically we can control the error for the independent variable by not allowing it to exceed a certain number. So this "largerst possible error" in the independent varible will result in the "maximum error" in the function. So in order to compute the "maximum error" in the function, we need to assume that the independent variable x achieves its largest possible error, i.e. $\Delta x =$ largest possible error.

(3) Linear approximation gives a good way to compute it

error $\Delta y = f(a+\Delta x) - f(a)$

but $f(a+\Delta x) \approx L(a+\Delta x) = f(a) + f'(a)\cdot \Delta x$

so $\Delta y \approx (f(a)+f'(a)\cdot\Delta x) - f(a) = f'(a)\cdot \Delta x$

remember that dx and Δx are the same thing.

denote $f'(a)\cdot \Delta x = f'(a)\cdot dx$ by "dy".

So "dy" gives an approximation of the maximum error in the function, which can be computed by $f'(a)\cdot dx$, where $f'(a)$ is the value of the derivative at the reference point, and dx is the largest possible error in x.

(4) relative error.

define the ratio of the actual error and the value of the function at the reference point to be the relative error

i.e. $\frac{\Delta y}{y}$, which can be approximated by $\frac{dy}{y} = \frac{f'(a)\cdot dx}{f(a)}$

Explain Example 4.

P283 29. (a) For a sphere of radius r, the circumference is $C = 2\pi r$ and the surface area is $S = 4\pi r^2$, so $r = \frac{C}{2\pi} \Rightarrow$

$S = 4\pi\cdot\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{\pi}$ so the function $f(C) = \frac{C^2}{\pi}$

From the problem, reference point $C = 84$,

largest possible error $dC = 0.5$,

$f'(C) = \frac{2C}{\pi} \Rightarrow f'(84) = \frac{2\times 84}{\pi}$

so $\Delta S \approx dS = f'(84) \cdot dS = \frac{2\times 84}{\pi} \cdot 0.5 = \frac{84}{\pi} \approx 27\ cm^2$

Relative error $= \frac{\Delta S}{S} \approx \frac{dS}{S} = \frac{\frac{84}{\pi}}{\frac{84^2}{\pi}} = \frac{1}{84} \approx 0.012$

Note: the unit of the error is the same as the unit of the function.
the relative error has no unit.

(b) $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{c}{2\pi}\right)^3 = \frac{c^3}{6\pi^2}$ so the function $g(c) = \frac{c^3}{6\pi^2}$

error $\Delta V \approx dV = g'(84) \cdot dc$

$g'(c) = \frac{3c^2}{6\pi^2} = \frac{c^2}{2\pi^2} \Rightarrow g'(84) = \frac{84^2}{2\pi^2}$

$dc = 0.5$

So $\Delta V \approx dV = \frac{84^2}{2\pi^2} \cdot 0.5 = \frac{1764}{\pi^2} \approx 179\ cm^3$

Relative error $\frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{\frac{1764}{\pi^2}}{\frac{84^3}{6\pi^2}} = \frac{1}{56} \approx 0.018$