Section Notes 10/23/07

Summary of computation of derivatives :

1. Basic properties :

$(c\cdot f(x))' = c\cdot f'(x)$

$(f(x)\pm g(x))' = f'(x)\pm g'(x)$

$(\text{constant})' = 0$

$(x^n)' = n\cdot x^{n-1}$ (power rule)

$(fg)' = f'g + fg'$ (product rule)

$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ (quotient rule)

$(f(g(x)))' = f'(g(x))\cdot g'(x)$ (chain rule)

2. Implicit differentiation :

If the function y (in terms of x) is given implicitly by an equation without being solved, use implicit differentiation.

Method : differentiate both sides of the equation as normal and every time you differentiate a "y" you tack on a "y'" (from the chain rule), then solve for y'

Important derivatives derived from implicit differentiation :

Inverse trigonometric functions, log functions

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$, $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$, $(\arctan x)' = \frac{1}{1+x^2}$

$(\log_a x)' = \frac{1}{x\ln a}$, $(\ln x)' = \frac{1}{x}$. $(\ln|x|)' = \frac{1}{x}$.

3. Logrithmic differentiation:

It's extremely useful in following 2 cases :

(1) the function is the product / quotient of lots of factors

(2) $(f(x))^{g(x)}$ where $f(x)$, $g(x)$ are functions of x.

Step 1 : take "ln" function on both sides, use law of Logrithms

Step 2 : differentiate implicitly with respect to x

Step 3 : Solve for y'

4. general principle when differentiating a function:

I. If in an explicit form $y=f(x)$, fit it into one of the following categories:

(1) one of the common derivatives you have to remember $\longrightarrow$ ✓

(2) sum / difference of two functions $\longrightarrow (f \pm g)' = f' \pm g'$

(3) product of two functions $\longrightarrow (fg)' = f'g + fg'$

(4) quotient of two functions $\longrightarrow \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

(5) composition of two or more functions $\longrightarrow (f(g(x)))' = f'(g(x))\, g'(x)$

(6) a function raised to the power of another function $\rightarrow$ Log diff

(7) exception: in (3) and (4), if there are lots of factors, one can try Log diff

II. If function given in an implicit form, use implicit differentiation

5. common derivatives

(1) $(c)' = 0$ where c is a constant, $(x^n)' = nx^{n-1}$

(2) $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\tan x)' = \sec^2 x$

$(\cot x)' = -\csc^2 x$, $(\sec x)' = \sec x \tan x$, $(\csc x)' = -\csc x \cot x$

(3) $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$, $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$, $(\tan^{-1} x)' = \frac{1}{1+x^2}$

(4) $(a^x)' = a^x \ln a$, in particular, $(e^x)' = e^x$.

(5) $(\log_a x)' = \frac{1}{x \ln a}$ (where $x>0$), in particular, $(\ln x)' = \frac{1}{x}$ (where $x>0$)

however, $(\ln|x|)' = \frac{1}{x}$ (where $x \neq 0$)

Examples:

P238, 9. implicit function $\rightarrow$ use implicit differentiation

take derivatives with respect to x on both sides

$\frac{d}{dx}(4\cos x \sin y) = \frac{d}{dx} 1$

RHS $= 0$

LHS: product $\rightarrow$ product rule

$$LHS = \frac{d}{dx}(4\cos x \sin y) = 4\frac{d}{dx}(\cos x \sin y)$$

$$= 4\left(\frac{d}{dx}\cos x\right)\cdot \sin y + 4\cos x\cdot\left(\frac{d}{dx}\sin y\right)$$

$$= 4(-\sin x)\sin y + 4\cos x \cdot \sin y \cdot y'$$

so we have

$$-4\sin x \sin y + 4\cos x \sin y\, y' = 0$$

Solve for y': $\quad y' = \dfrac{4\sin x \sin y}{4\cos x \cos y} = \tan x \cdot \tan y$

P238, 15 $\quad x^2+xy+y^2=3$

(1) tangent line at $(1,1)$?

(2) at which pts does the curve have a horizontal tangent ?

In both problems, need slopes, i.e. derivatives

implicit differentiation:

$$\frac{d}{dx}(x^2+xy+y^2) = \frac{d}{dx}3$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}xy + \frac{d}{dx}y^2 = 0$$

$$2x + (y+xy') + 2yy' = 0$$

$$(2x+y)+(x+2y)y' = 0$$

$$y' = -\frac{2x+y}{x+2y}$$

(1) at $(1,1)$, slope $= y' = -\frac{3}{3} = -1$

so tangent line: $y-1 = -(x-1) \Rightarrow y = -x+2$

(2) horizontal tangents: slope $= y' = 0$

$$\Rightarrow -\frac{2x+y}{x+2y} = 0 \Rightarrow 2x+y=0$$

so points on the curve with horizontal tangents

= points on the curve which satisfy $2x+y=0$

= points which are common solutions of $\begin{cases} x^2+xy+y^2=3 & ① \\ 2x+y=0 & ② \end{cases}$

Solve this system for pairs (x, y)

from ②: $y = -2x$ ③

plug in ③ to ①: $x^2 + x(-2x) + (-2x)^2 = 3$

$\Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

corresponding y-coordinates: $y = -2x = \mp 2$

so at points $(1, -2)$, $(-1, 2)$, the curve has horizontal tangents

P239. 35. $y = \arcsin(\tan\theta)$

explicit function, composition of two functions $\Rightarrow$ chain rule

inner function: $u = \tan\theta$ $\Rightarrow \frac{du}{d\theta} = \sec^2\theta$

outer function: $y = \arcsin u$ $\Rightarrow \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$

by chain rule, $y' = \frac{1}{\sqrt{1-\tan^2\theta}} \cdot \sec^2\theta$

P245. 17. $y = \ln(e^{-x} + xe^{-x})$

explicit function, composition of two functions: $\Rightarrow$ chain rule

inner function: $u = e^{-x} + xe^{-x}$

outer function: $y = \ln u$

we need derivatives of the two functions to apply chain rule.

$\frac{dy}{du} = \frac{1}{u} = \frac{1}{e^{-x}+xe^{-x}}$

$\frac{du}{dx} = \frac{d}{dx}\underbrace{(e^{-x} + xe^{-x})}_{\text{sum}} = \frac{d}{dx}\underbrace{(e^{-x})}_{\text{composition}} + \frac{d}{dx}\underbrace{(xe^{-x})}_{\text{product}}$

$\frac{d}{dx}(e^{-x}) = e^{-x} \cdot (-1)$ (chain rule!)

$\frac{d}{dx}(xe^{-x}) = \frac{d}{dx}x \cdot e^{-x} + x\frac{d}{dx}e^{-x}$ (product rule!)

$= 1 \cdot e^{-x} + x \cdot e^{-x} \cdot (-1)$ (chain rule!)

So $\frac{du}{dx} = -e^{-x} + e^{-x} - x \cdot e^{-x} = -xe^{-x}$

by chain rule, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{e^{-x}+xe^{-x}} \cdot (-xe^{-x}) = \frac{-x}{1+x}$

P246 29. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}$

explicit function, product / quotient of lots of factors $\Rightarrow$ log. diff.

take "ln" on both sides:

$$\ln y = \ln \frac{\sin^2 x \tan^4 x}{(x^2+1)^2} = 2\ln(\sin x) + 4\ln(\tan x) - 2\ln(x^2+1)$$

differentiate implicitly:

$$\frac{1}{y} y' = 2\frac{1}{\sin x}\cos x + 4\frac{1}{\tan x}\cdot \sec^2 x - 2\cdot\frac{1}{x^2+1}\cdot 2x \quad \text{(chain rule!)}$$

$$\Rightarrow \quad y' = y\cdot\left(2\frac{\cos x}{\sin x} + 4\frac{\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right)$$

$$= \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(\frac{2\cos x}{\sin x} + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right)$$

P246 41. want to compute the limit $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}$.

realize it as the derivative of a function by definition.

remember: 2 formulae for definition of derivative:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{or} \quad f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

You can use either one

Approach 1: If use the first one, observe $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x} = \lim\limits_{h\to 0}\dfrac{\ln(1+h)}{h}$

$$= \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h}$$

So $a=1$, $f(x) = \ln x$ and the limit is $f'(1)$

However, $f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1$

So the limit equals 1

Approach 2: If use the second definition, you may get a different function and a different point, but the value of the derivative should be the same as in the previous approach.

P246 33. $y = (\cos x)^x$

explicit function, $(f(x))^{g(x)}$ form $\to$ log diff.

take "ln" on both sides:

$$\ln y = \ln((\cos x)^x) = x \ln(\cos x)$$

differentiate implicitly

$$(LHS)' = \frac{1}{y} \cdot y' \quad \text{(chain rule)}$$

$$(RHS)' = 1 \cdot \ln(\cos x) + x \cdot (\ln(\cos x))' \quad \text{(product rule)}$$

$$= \ln(\cos x) + x \frac{1}{\cos x} \cdot (-\sin x) \quad \text{(chain rule)}$$

so $\frac{1}{y} \cdot y' = \ln(\cos x) - \frac{x \sin x}{\cos x}$

$$y' = y \cdot (\ln(\cos x) - x \tan x)$$

$$= (\cos x)^x (\ln(\cos x) - x \tan x)$$