Section Notes 10/09/07

$\boxed{1}$. Asymptotes.

Horizontal asymptotes: $y=L$ is a horizontal asymptote if $\lim_{x\to(+\text{or}-)\infty} f(x)=L$

Vertical asymptotes: $x=a$ is a vertical asymptote if $\lim_{x\to a(\pm)} f(x)=(+\text{ or }-)\infty$

P137 Problem 3 (f)

$\lim_{x\to\infty} f(x)=1$ $\lim_{x\to-\infty} f(x)=2 \Rightarrow$ horizontal asymptotes $y=1$, $y=2$

+ or − infinity as limit or one-sided limit at −1, 2

$\Rightarrow$ vertical asymptotes $x=-1$, $x=2$

$\boxed{2}$ graph sketching problems.

P139. problem 41

(1) Sketch a graph

Step 1. group the conditions according to the value of x.

at $-\infty$: $\lim_{x\to-\infty} f(x)=\infty$

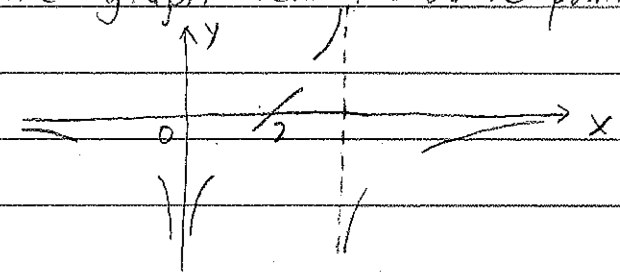
at 0 : $\lim_{x\to 0} f(x)=-\infty$

at 2 : $f(2)=0$

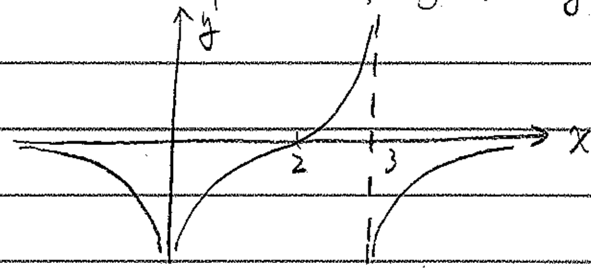
at 3 : $\lim_{x\to 3-} f(x)=+\infty$, $\lim_{x\to 3+} f(x)=-\infty$

at $+\infty$: $\lim_{x\to+\infty} f(x)=0$

step 2. Sketch the graph near the above points



Step3. Connect the little pieces of graph by smooth curves



(2) give a possible formula for $f(x)$. Let's look for a rational function.
note the following information:

$\lim_{x\to\pm\infty} f(x)=0 \Rightarrow$ degree of numerator < degree of denominator

$\lim_{x\to 3^-} f(x)=\infty$, $\lim_{x\to 3^+} f(x)=-\infty \Rightarrow x-3$ in denominator

$\lim_{x\to 0} f(x)=-\infty \Rightarrow x$ in denominator, actually we need x^2 in denominator because when x passes through 0 from negative to positive, all other factors in our expression won't have their sign changed, except the factors of x. If there's only one "x" in the denominator, when x passes through 0, the sign of $f(x)$ will also change. This explains why we need a double factor of x, or say, x^2

$f(2)=0 \Rightarrow x-2$ in numerator

Combine all the above information, we should try: $\dfrac{x-2}{x^2(x-3)}$

If we compute $\lim_{x\to 3^-} f(x)$, we get: $\dfrac{\overset{\to 1}{x-2}}{\underset{\downarrow 3^2}{x^2}\underset{\searrow 0^-}{(x-3)}} \to -\infty$.

while we need the limit to be ∞

so we need an extra "$-$" sign, therefore $f(x)=-\dfrac{x-2}{x^2(x-3)}$ is a possible answer. You can check it satisfies all the above conditions.

$\boxed{3}$ limit computations

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15. $\lim\limits_{x\to -3+} \dfrac{x+2}{x+3} = -\infty.$ ($x+2 \to (-1)$, $x+3 \to (0+)$)

19 $\lim\limits_{x\to 3+} \ln(x^2-9)$ $x\to 3+ \Rightarrow x^2 \to 9+ \Rightarrow x^2-9 \to 0+$.

so $\lim\limits_{x\to 3+} \ln(x^2-9) = \lim\limits_{t\to 0+} \ln t = -\infty$

Recall: $\lim\limits_{x\to 0+} \ln x = -\infty$, $\lim\limits_{x\to -\infty} e^x = 0$

21. $\lim\limits_{x\to \infty} \dfrac{x^3+5x}{2x^3-x^2+4}$ when $x\to\infty$.

principle: Create $\frac{1}{x}$ because $\lim\limits_{x\to\infty} \frac{1}{x} = 0$.

method: divide numerator & denominator by the dominant power (highest power)

in this problem, dominant power: x^3.

$$\lim_{x\to\infty} \frac{x^3+5x}{2x^3-x^2+4} = \lim_{x\to\infty} \frac{\frac{x^3+5x}{x^3}}{\frac{2x^3-x^2+4}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{5}{x^2}}{2-\frac{1}{x}+\frac{4}{x^3}}$$

$$= \frac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty} \frac{5}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty} \frac{1}{x} + \lim\limits_{x\to\infty} \frac{4}{x^3}} = \frac{1+0}{2-0+0} = \frac{1}{2}$$

Remark:

$$\lim_{x\to\infty} \frac{a_0 x^m + a_1 x^{m-1} + \cdots}{b_0 x^n + b_1 x^{n-1} + \cdots} = \begin{cases} 0 & m<n \\ \frac{a_0}{b_0} & m=n \\ (+\text{ or }-)\infty & m>n \end{cases}$$

(depend on the sign of $\frac{a_0}{b_0}$)

(rational function with x^m, x^n as the highest power in the numerator / denominator)

Exercise: prove it by using the above method.

25 $\lim_{x\to\infty} \left(\sqrt{9x^2+x} - 3x\right)$ Idea: Indeterminate type $\infty - \infty$, "$\sqrt{\ }$" suggests "rationalizing".

$= \lim_{x\to\infty} \dfrac{(\sqrt{9x^2+x} - 3x)(\sqrt{9x^2+x}+3x)}{\sqrt{9x^2+x}+3x}$

$= \lim_{x\to\infty} \dfrac{(9x^2+x)-(3x)^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \dfrac{x}{\sqrt{9x^2+x}+3x}$

Second Idea: in order to create $\frac{1}{x}$, divide top/bottom by the dominate power x.

$= \lim_{x\to\infty} \dfrac{\frac{x}{x}}{\frac{\sqrt{9x^2+x}+3x}{x}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{9+\frac{1}{x}}+3}$

$= \dfrac{\lim_{x\to\infty} 1}{\sqrt{\lim_{x\to\infty} 9 + \lim_{x\to\infty}\frac{1}{x}} + \lim_{x\to\infty} 3} = \dfrac{1}{\sqrt{9+0}+3} = \dfrac{1}{6}$

29 $\lim_{x\to\infty} e^{-2x}\sin x$ principle: when sin/cosine is involved, and substitution rule does not apply, use squeeze theorem

$-1 \le \sin x \le 1 \Rightarrow -e^{-2x} \le e^{-2x}\sin x \le e^{-2x}$

$\lim_{x\to\infty} e^{-2x} = 0$, $\lim_{x\to\infty}(-e^{-2x}) = 0 \Longrightarrow \lim_{x\to 0} e^{-2x}\sin x = 0$ by squeeze theorem.

$\boxed{4}$ rate of change in application problems

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25. (a) (i) $[2000, 2002]$: $\dfrac{P(2002)-P(2000)}{2002-2000} = \dfrac{77-55}{2} = 11$ $\underbrace{\text{percent/year}}_{\text{unit}}$.

(ii) and (iii) do them on your own.

(b) take the average of the values from (ii) and (iii)

in (ii), we have 13 percent/year

in (iii), we have 16 percent/year

so averagely, $\dfrac{13+16}{2} = 14.5$ percent/year

27. (a) (i) average rate of change

$$\frac{\Delta C}{\Delta x} = \frac{C(105)-C(100)}{105-100} = \cdots = 20.25 \text{ dollars/unit}$$

(do the computation by yourself)

(ii) do it on your own. answer: 20.05 dollars/unit.

(b) recall the instantaneous rate of change of a function $f(x)$ at a point "a" is given by $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ (where "a" is a fixed number)

so in this problem, it's

$$\lim_{h\to 0} \frac{C(100+h)-C(100)}{h} = \lim_{h\to 0} \frac{(5000+10(100+h)+0.05(100+h)^2)-(5000+10\cdot 100+0.05\cdot 100^2)}{h}$$

$$= \cdots = \lim_{h\to 0} \frac{20h+0.05h^2}{h} = \lim_{h\to 0}(20+0.05h) = 20 \text{ dollars/unit}$$

(always include the unit in an application problem)

Remark: it's also OK if you use the formula $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ to compute the instantaneous rate of change.
However, if you use this formula, your computation may be more involved. But you will definitely end up with the same answer..

$\boxed{5}$ derivatives

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21. You have two formulae to use:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Note that in both formulae, a is a fixed number.
In the first formula, the variable h always approaches 0.
But in the second formula, the variable x approaches a, which is not necessarily 0. So if you see the variable approaches a non-zero number, you should always use the second formula; while if you see the variable approaches 0, you can use both.
In this problem, the variable approaches 5 which is not 0, so you should use the second formula. Compare the second formula with the given expression, it's not hard to see that $a=5$ and $f(x)=2^x$
If you are still feeling confused, try to do problems 19-24 on the same page.

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29. (a) $f'(v)$ is the rate at which the fuel consumption is changing with respect to the speed. Its unit is the ratio of the units of fuel consumption and the speed, therefore is $\frac{gal/h}{mi/h}$

(b) the fuel consumption is decreasing by $0.05\ \frac{gal/h}{mi/h}$ as the car's speed reaches $20\ mi/h$. So if you increase your speed to $21\ mi/h$, you could expect to decrease your fuel consumption by about $0.05\ gal/h$