

DIFFERENTIALS AND ERRORS

CONTENTS

1. Differentials	1
2. Errors	2
3. Applications	2

1. DIFFERENTIALS

Given a function $y = f(x)$, we call dx and dy differentials corresponding to the two variables and they have a very simple relation. In order to find this relation, remember that we have two notations for derivatives, that is,

$$f'(x) = \frac{dy}{dx},$$

which says the ratio of the two differentials is actually the derivative of the function. So the relationship between the two differentials can be given by

$$dy = f'(x)dx.$$

Note that if we are just given the function $f(x)$ then the differentials can be written as dx and df and we compute them in the same manner:

$$df = f'(x)dx.$$

Let's compute a couple of differentials and then explain what they mean.

Example 1. *Find the relation between the differentials for each of the following functions.*

- (1) $y = t^3 - 4t^2 + 7t$;
- (2) $w = x^2 \sin(2x)$;
- (3) $f(z) = e^{3-z^4}$.

Solution. As long as you remember that the differentials are related by the derivative as a ratio, there's not much to do here other than taking a derivative, and don't forget to add on the second differential to the derivative. Here are the solutions:

- (1) $dy = (3t^2 - 8t + 7)dt$;
- (2) $dw = (2x \sin(2x) + 2x^2 \cos(2x))dx$ (product rule and chain rule);
- (3) $df = -4z^3 e^{3-z^4} dz$ (chain rule).

□

by Ziyu Zhang, Oct 29, 2007. This note heavily relies on some online resources and is only used for teaching purpose.

2. ERRORS

Now let's switch to errors. They will be related to the differentials later.

If we think of Δx as the change in x , then $\Delta y = f(x + \Delta x) - f(x)$ is the change in y corresponding to the change in x . If Δx is small, then Δy is small as well. In this case, we sometimes call them "errors" in x and y respectively. So, remember, the "errors" measure changes in x and y .

We also think of the differential dx the same as Δx , that is, change in x , or error in x , so we can write $\Delta x = dx$. But since dy and Δy are defined in two different ways, they are in general not equal. The good thing is, when Δx is very small, they are very close to each other. That is to say, dy gives a very good approximation to Δy . So we can write $\Delta y \approx dy$. Let's see an illustration of the idea.

Example 2. Compute dy and Δy if $y = \cos(x^2 + 1) - x$ as x changes from $x = 2$ to $x = 2.03$.

Solution. First let's compute actual change in y , that is Δy .

$$\Delta y = (\cos(2.03^2 + 1) - 2.03) - (\cos(2^2 + 1) - 2) \approx 0.083581127.$$

Now let's get the formula for dy . As you know, it's no more than computing the derivative.

$$dy = (-2x \sin(x^2 + 1) - 1)dx.$$

Next, the change in x from $x = 2$ to $x = 2.03$ is $\Delta x = 0.03$. As what is explained above, dx is always the same as Δx , so we have $dx = \Delta x = 0.03$. This gives a value for the differential dy by

$$dy = (-2 \cdot 2 \sin(2^2 + 1) - 1) \cdot 0.03 \approx 0.085070913.$$

We can see that in fact we do have that $\Delta y \approx dy$ provided we keep Δx small. □

What is the reason for $\Delta y \approx dy$? The essence behind it is the linear approximation. Actually, Δy consists of many parts, in which dy is the linear part of change in y . However, compared to this linear part, the other parts of Δy are very very small. So the value of Δy is almost the same as the value of dy . This is the actual meaning of the differential dy and why it can be used to approximate Δy .

3. APPLICATIONS

We can use the fact that $\Delta y \approx dy$ in the following way.

Example 3 (Example 4, on page 251). A sphere was measured and its radius was found to be 21cm with a possible error of no more than 0.05cm. Estimate the maximum possible error in the volume if we use this value of the radius.

Solution. First, recall the equation for the volume of a sphere

$$V = \frac{4}{3}\pi r^3.$$

Now, if we start with $r = 21$ and use $dr = \Delta r = 0.05$, then $\Delta V \approx dV$ should give us maximum error. So, first get the formula for the differential

$$dV = 4\pi r^2 dr.$$

Now compute dV as

$$\Delta V \approx dV = 4\pi \cdot 21^2 \cdot 0.05 \approx 277\text{cm}^3,$$

then the maximum error in the volume is then approximately 277cm^3 . \square

Remark. Be careful not to assume this is a large error. On the surface it looks large, however if we compute the actual volume for $r = 21\text{cm}$, we get

$$V = \frac{4}{3}\pi \cdot 21^3 \approx 38792\text{cm}^3.$$

So, in comparison the error in the volume is:

$$\frac{\Delta V}{V} \approx \frac{277}{38792} \approx 0.007 = 0.7\%.$$

Here 0.007 is called **relative error** and 0.7% is called **percentage error**. They mean that the error in the volume is no more than 0.007 times the actual volume (or 0.7% of the actual volume, respectively). So we still see that the error is very small. \square

Example 4 (Problem 29, part (a), on page 253). *The circumference of a sphere was measured to be 84cm with a possible error of 0.5cm . Use differentials to estimate the maximum error in the calculated surface area. What is the relative error?*

Outline. We will follow these steps to do this problem:

Step 1. find the function of surface area in terms of circumference;

Step 2. compute the differential by differentiating the function;

Step 3. use the differential to estimate the error;

Step 4. estimate the relative error.

Solution. According the above steps, we have:

Step 1. Compute the function.

For a sphere of radius r , we have

circumference $C = 2\pi r$;

surface area $S = 4\pi r^2$.

If we want to express S as a function of C , from the expression of the circumference we have $r = \frac{C}{2\pi}$, therefore

$$S = 4\pi\left(\frac{C}{2\pi}\right)^2 = \frac{C^2}{\pi}.$$

Step 2. Compute the differential.

$S = \frac{C^2}{\pi}$ implies

$$dS = \frac{2C}{\pi}dC$$

Step 3. Estimate the error.

We have $C = 84$ and $dC = \Delta C = 0.5$, so error in S is

$$\Delta S \approx dS = \frac{2 \cdot 84}{\pi} \cdot 0.5 = \frac{84}{\pi} \approx 27\text{cm}^2.$$

Step 4. Estimate the relative error.

We have

$$S = \frac{C^2}{\pi} = \frac{84^2}{\pi}.$$

So relative error is

$$\frac{\Delta S}{S} = \frac{\frac{84}{\pi}}{\frac{84^2}{\pi}} = \frac{1}{84} \approx 0.012.$$

The conclusion is: the maximum error in surface area is approximately 27cm^2 , and the relative error is approximately 0.012. \square

Remark. Remember to add on the unit for the error, which is the same as the unit of the function. The relative error doesn't have a unit. \square

Example 5 (Problem 29, part (b), on page 253). *The circumference of a sphere was measured to be 84cm with a possible error of 0.5cm . Use differentials to estimate the maximum error in the calculated volume. What is the relative error?*

Solution. We follow the same steps as in the previous problem.

Step 1. Find the function of volume in terms of circumference.

For a sphere of radius r ,

circumference $C = 2\pi r$,

volume $V = \frac{4}{3}\pi r^3$.

From the formula for the circumference we have $r = \frac{C}{2\pi}$, therefore

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \left(\frac{C}{2\pi}\right)^3 = \frac{C^3}{6\pi^2}.$$

Step 2. Find the differential.

$$dV = \frac{3C^2}{6\pi^2}dC = \frac{C^2}{2\pi^2}dC.$$

Step 3. Estimate the error.

From the given conditions, $C = 84$ and $dC = \Delta C = 0.5$. So the error

$$\Delta V \approx dV = \frac{84^2}{2\pi^2} \cdot 0.5 = \frac{1764}{\pi^2} \approx 179\text{cm}^3.$$

Step 4. Estimate the relative error.

$$V = \frac{C^3}{6\pi^2} = \frac{84^3}{6\pi^2}.$$

So the relative error

$$\frac{\Delta V}{V} = \frac{\frac{1764}{\pi^2}}{\frac{84^3}{6\pi^2}} = \frac{1}{56} \approx 0.018.$$

The conclusion is: the maximum error in volume is approximately 179cm^3 , and the relative error is approximately 0.018. \square