Solution to Problem 53 on Page 229*

Analysis and Solution For the tangent line to be horizontal, we want the slope of the tangent to be 0. The slope of the tangent can be explained as the derivative of the function. So actually we want

$$f'(x) = 0$$

Here we have

$$f(x) = 2\sin x + \sin^2 x,$$

therefore,

$$f'(x) = 2\cos x + 2\sin x\cos x$$

(note that we need the chain rule for the second term). So we want to solve the equation

$$2\cos x + 2\sin x\cos x = 0.$$

Factor out $\cos x$ and we get

$$2\cos x(1 + \sin x) = 0$$

So we want either $\cos x = 0$ or $1 + \sin x = 0$.

Now we have two choices. If $\cos x = 0$, since the cosine function is a periodic function with the minimal positive period 2π, we can first solve the equation in one single period, then get all roots by adding integer multiples of the period 2π. In a single period, say $[0, 2\pi)$, the values which make the cosine function to be 0 are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$, so all the roots of $\cos x = 0$ are

$$x = \frac{\pi}{2} + 2n\pi$$

and

$$x = \frac{3\pi}{2} + 2n\pi$$

where n is any integer. Note that we start by considering the roots in $[0, 2\pi)$ just because it is the simplest single period of the cosine function. You can actually use any half closed half open interval of length 2π.

Similarly, if we want $1 + \sin x = 0$, that is $\sin x = -1$, we first solve it in a single period $[0, 2\pi)$. The only number in this interval which gives $\sin x = -1$ is $x = \frac{3\pi}{2}$. So by adding integer multiples of 2π, we get all roots of the equation $1 + \sin x = 0$ which are

$$x = \frac{3\pi}{2} + 2n\pi$$

*by Ziyu Zhang, Oct 27, 2005, revised at Oct 18, 2007 where n is any integer.

Combine the above two cases (by taking the union of all roots in the two cases and eliminating the redundant ones), and we find that the roots of the function

$$f'(x) = 2\cos x + 2\sin x \cos x = 0$$

are

$$x = \frac{\pi}{2} + 2n\pi$$

and

$$x = \frac{3\pi}{2} + 2n\pi$$

where n is any integer.

Finally the problem asks us to find out the points where the tangents are horizontal. We have already found the x-coordinates of these points. In order to find out the y-coordinates, we can just plug in the values of x into the original function $f(x)$ and calculate the corresponding values of y.

When $x = \frac{\pi}{2} + 2n\pi$ where n is any integer, because of the periodicity of the sine function, we know that

$$\sin x = \sin\left(\frac{\pi}{2} + 2n\pi\right) = \sin\frac{\pi}{2} = 1$$

hence

$$y = 2\sin x + \sin^2 x = 2 \cdot 1 + 1^2 = 3$$

Similarly, when $x = \frac{3\pi}{2} + 2n\pi$ where n is any integer, we have that

$$\sin x = \sin\left(\frac{3\pi}{2} + 2n\pi\right) = \sin\frac{3\pi}{2} = -1$$

hence

$$y = 2\sin x + \sin^2 x = 2 \cdot (-1) + (-1)^2 = -1$$

So the points on the curve with horizontal tangents are $(\frac{\pi}{2} + 2n\pi, 3)$ and $(\frac{3\pi}{2} + 2n\pi, -1)$, where n is any integer.

Feel free to contact me if you have any questions about it.