Solutions to Problems in Appendix D[*]

- **Problem 1 of Appendix D, Page A37**

  Please take a careful look at the graph given in the problem. We need to find δ such that $|\frac{1}{x} - 0.5| < 0.2$ whenever $|x-2| < \delta$.

  Note that in order to make the difference between $\frac{1}{x}$ and 0.5 less than 0.2, it is required that $0.3 < \frac{1}{x} < 0.7$. By looking at the graph, we see that this condition is satisfied when x satisfies $\frac{10}{7} < x < \frac{10}{3}$. So on the left side of $x = 2$, we need the distance between x and 2 to be less than the distance between $\frac{10}{7}$ and 2, i.e. $|x-2| < |\frac{10}{7} - 2| = \frac{4}{7}$. Similarly, on the right side of $x = 2$, we need $|x - 2| < |\frac{10}{3} - 2| = \frac{4}{3}$. Since we want to find a single δ to make both conditions satisfied at once (we want $|x-2| < \frac{4}{7}$ and $|x-2| < \frac{4}{3}$), we need the more restrictive of the two to hold. Since $\frac{4}{7} < \frac{4}{3}$, the condition $|x-2| < \frac{4}{7}$ is more restrictive than $|x-2| < \frac{4}{3}$, therefore we only need to make sure that by choosing a suitable δ, whenever $|x-2| < \delta$, the inequality $|x-2| < \frac{4}{7}$ is satisfied. So clearly we can choose $\delta = \frac{4}{7}$, or any smaller positive number.

- **Problem 9 of Appendix D, Page A37**

  Preliminary Analysis: Recall the following definition: $\lim_{x\to a} f(x) = L$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$.

  So let ε be an arbitrary positive number. We need to find another positive number δ (i.e. express δ in terms of ε), such that $|(3x-2)-4| < \varepsilon$ whenever $|x-2| < \delta$.

  But $|(3x - 2) - 4| = |3x - 6| = 3|x - 2|$, so we want $3|x - 2| < \varepsilon$ whenever $|x - 2| < \delta$. But $3|x - 2| < \varepsilon$ is the same as $|x - 2| < \frac{\varepsilon}{3}$ (we

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just divided both sides by 3), so actually we want $|x-2| < \frac{\varepsilon}{3}$ whenever $|x-2| < \delta$. Hence if we take $\delta = \frac{\varepsilon}{3}$, or any positive number less than $\frac{\varepsilon}{3}$, then for any x satisfying $|x-2| < \delta$, we do have $|x-2| < \frac{\varepsilon}{3}$. The argument above shows that we can choose $\delta = \frac{\varepsilon}{3}$ or any other smaller positive number.

Proof: For any given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{3}$. For any x satisfying $|x-2| < \delta$, we have $|(3x-2)-4| = |3x-6| = 3|x-2| < 3\delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$. So we've shown that $|(3x-2)-4| < \varepsilon$ whenever $|x-2| < \delta$. By the definition of a limit, we have $\lim_{x\to 2}(3x-2) = 4$.

Remark: In addition to studying these solutions, I would also recommend studying examples 3 in Appendix D (page A30-A33). If you have more questions about these two problems, or any questions about the material from Appendix D, please let me know.

- **Some more exercises for you to practise on your own**

  1. Show that $\lim_{x\to 2}(3x-7) = -1$ by finding a $\delta > 0$ such that
  $$|(3x-7)-(-1)| < \varepsilon$$
  whenever $0 < |x-2| < \delta$.
  2. Show that $\lim_{x\to 0}(-2x+5) = 5$ by finding a $\delta > 0$ such that
  $$|(-2x+5)-5| < \varepsilon$$
  whenever $0 < |x-0| < \delta$.
  3. Show that $\lim_{x\to -1}(4x+3) = -1$ by finding a $\delta > 0$ such that
  $$|(4x+3)-(-1)| < \varepsilon$$
  whenever $0 < |x-(-1)| < \delta$.