Applications of Intermediate Value Theorem*

If a function $f(x)$ is continuous on a closed interval $[a, b]$, all the Intermediate Value Theorem is really saying is that a continuous function will take on all values between $f(a)$ and $f(b)$.

Look at figure 8 on page 124 in the textbook. If we pick any value, N, that is between the value of $f(a)$ and the value of $f(b)$ and draw a line straight out from this point the line will hit the graph in at least one point. In other words somewhere between a and b the function will take on the value of N. Also, as the figure shows the function may take on the value at more than one place.

It's also important to note that the Intermediate Value Theorem only says that the function will take on the value of N somewhere between a and b. It doesn't say just what that value will be. It only says that it exists.

So, the Intermediate Value Theorem tells us that a function will take the value of N somewhere between a and b but it doesn't tell us where it will take the value nor does it tell us how many times it will take the value. These are important ideas to remember about the Intermediate Value Theorem.

A nice application of the Intermediate Value Theorem is to prove the existence of roots of equations as the following example shows.

- **Problem 39 of Section 2.4, Page 127**

  As the problems says, we want to use the Intermediate Value Theorem to show that the equation $\cos x = x$ has a root between 0 and 1.

  Step 1. Create a new function by subtracting the right hand side of the equation from the left hand side, i.e.

$$f(x) = \cos x - x.$$

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By doing this we come up with a new problem which is equivalent to the original one: is there a value c for x between 0 and 1 such that $f(c) = 0$?

In order to apply Intermediate Value Theorem, we need to verify that the two conditions in the theorem are satisfied, namely the function $f(x)$ is continuous in the specified closed interval $[0, 1]$, and the value 0 is between $f(0)$ and $f(1)$.

Step 2. Since trigonometric functions and power functions are both continuous, $\cos x$ and x are both continuous, so $f(x)$, as the difference of two continuous functions, is also continuous in its domain, which is all real numbers. In particular, it is continuous in the interval $[0, 1]$.

Step 3. Calculate the values $f(0)$ and $f(1)$, and compare them with 0. Obviously,

$$f(0) = \cos(0) - 0 = 1 - 0 = 1 > 0,$$

$$f(1) = \cos(1) - 1 < 0.$$

In the above line we used the fact that the value of a cosine function is always no larger than 1 and since 1 is not an integer multiple of 2π, $\cos(1)$ is not equal to 1, therefore is strictly smaller than 1.

Step 4. Now we are ready to apply the Intermediate Value Theorem. Since $f(x)$ is continuous on the closed interval $[0, 1]$ and 0 is between $f(0)$ and $f(1)$, by Intermediate Value Theorem, we can find a number c between 0 and 1, such that $f(c) = 0$, that is to say, $\cos(c) - c = 0$. So the number c satisfies $\cos(c) = c$ which is what we want.

- **Problem 47 of Section 2.4, Page 127**

  If such a number (which we will denote by c) exists, then it must satisfy $c = c^3 + 1$, or $c^3 - c + 1 = 0$. So if we define a function $f(x) = x^3 - x + 1$, then c must be a root of the equation $x^3 - x + 1 = 0$. So now our problem is: is there a root for this equation?

  By virtue of the Intermediate Value Theorem, we can give a positive answer to this. $f(x)$ is clearly continuous because it is a polynomial (see theorem 5 of this section). So we only need to find an interval, such that the values of the function at the two endpoints of the interval have different signs. Then by the Intermediate Value Theorem, the equation $f(x) = 0$ has a root in this interval.

You can have many different choices. Here is one of these. Note that $f(-2) = (-2)^3 - (-2) + 1 = -8 + 2 + 1 = -5 < 0$ and $f(0) = 0^3 - 0 + 1 = 1 > 0$. So we have $f(-2) < 0 < f(0)$. Because of the continuity of $f(x)$, the Intermediate Value Theorem tells us that there is at least one c in the interval $(-2, 0)$ such that $f(c) = 0$, that is $c^3 - c + 1 = 0$, or $c = c^3 + 1$. Now we know that there does exist a number c such that c is exactly 1 more than c^3.

Remark: Also take a look at example 10 (page 125) which is very similar to the problems above.

Remark: In doing this type of problems, always remember to explicitly write down that your function is **continuous** in the specified closed interval. Also remember to mention the name of the theorem you are using. Actually it is a general rule that whenever you want to quote a theorem which has a concrete name, mention the name of the theorem in your solution.

- **Some more exercises for you to practise on your own**

  1. Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere between -1 and 2.
  2. Show that there is some u with $0 < u < 2$ such that $u^2 + \cos(\pi u) = 4$.
  3. Show that $\ln(x) = e^{-x}$ has a root between 1 and 2.
  4. Show that the equation $\arctan x = 1 - x$ has at least one real root. (Hint: you need to carefully choose your own interval as the second example above. Try a few simple values of x until you find two appropriate ones.)

Try to work on these examples. If you have any more questions, don't hesitate to ask me. Problems like the second examples above might be hard for some of you, but everyone should be able to do problems like the first example above which is very likely to show up in the midterm next week.