Mathematical Introduction to Data Science	**Mar 8, 2016**

Homework 2. Random Matrix Theory and Phase Transitions in PCA

Instructor: Yuan Yao — *Due: Tuesday Mar 15, 2016*

The problem below marked by * is optional with bonus credits.

1. *Phase transition in PCA "spike" model:* Consider a finite sample of n i.i.d vectors $x_1, x_2, \ldots, x_n$ drawn from the p-dimensional Gaussian distribution $\mathcal{N}(0, \sigma^2 I_{p\times p} + \lambda_0 uu^T)$, where λ_0/σ^2 is the signal-to-noise ratio (SNR) and $u \in \mathbb{R}^p$. In class we showed that the largest eigenvalue λ of the sample covariance matrix S_n

$$S_n = \frac{1}{n}\sum_{i=1}^{n} x_i x_i^T$$

pops outside the support of the Marcenko-Pastur distribution if

$$\frac{\lambda_0}{\sigma^2} > \sqrt{\gamma},$$

or equivalently, if

$$\mathrm{SNR} > \sqrt{\frac{p}{n}}.$$

(Notice that $\sqrt{\gamma} < (1+\sqrt{\gamma})^2$, that is, λ_0 can be "buried" well inside the support Marcenko-Pastur distribution and still the largest eigenvalue pops outside its support). All the following questions refer to the limit $n \to \infty$ and to almost surely values:

   (a) Find λ given $\mathrm{SNR} > \sqrt{\gamma}$.

   (b) Use your previous answer to explain how the SNR can be estimated from the eigenvalues of the sample covariance matrix.

   (c) Find the squared correlation between the eigenvector v of the sample covariance matrix (corresponding to the largest eigenvalue λ) and the "true" signal component u, as a function of the SNR, p and n. That is, find $|\langle u, v\rangle|^2$.

   (d) Confirm your result using MATLAB or R simulations (e.g. set $u = e$; and choose $\sigma = 1$ and λ_0 in different levels. Compute the largest eigenvalue and its associated eigenvector, with a comparison to the true ones.)

2. *Exploring S&P500 Stock Prices:* Take the Standard & Poor's 500 data:
`http://math.stanford.edu/~yuany/course/data/snp452-data.mat`,
which contains the data matrix $X \in \mathcal{R}^{n\times p}$ of $n = 1258$ consecutive observation days and $p = 452$ daily closing stock prices, and the cell variable "stock" collects the names, codes, and the affiliated industrial sectors of the 452 stocks. Use Matlab or R for the following exploration.

(a) Take the logarithmic prices $Y = \log X$;

(b) For each observation time $t \in \{1, \ldots, 1257\}$, calculate logarithmic price jumps

$$\Delta Y_{i,t} = Y_{i,t} - Y_{i,t-1}, \quad i \in \{1, \ldots, 452\};$$

(c) Construct the realized covariance matrix $\hat{\Sigma} \in \mathcal{R}^{452 \times 452}$ by,

$$\hat{\Sigma}_{i,j} = \frac{1}{1257} \sum_{\tau=1}^{1257} \Delta Y_{i,\tau} \Delta Y_{j,\tau};$$

(d) Compute the eigenvalues (and eigenvectors) of $\hat{\Sigma}$ and store them in a descending order by $\{\hat{\lambda}_k, k = 1, \ldots, p\}$.

(e) *Horn's Parallel Analysis*: the following procedure describes a so-called Parallel Analysis of PCA using random permutations on data. Given the matrix $[\Delta Y_{i,t}]$, apply random permutations $\pi_i : \{1, \ldots, t\} \to \{1, \ldots, t\}$ on each of its rows: $\Delta \tilde{Y}_{i,\pi_i(j)}$ such that

$$[\Delta \tilde{Y}_{\pi(i),t}] = \begin{bmatrix} \Delta Y_{1,1} & \Delta Y_{1,2} & \Delta Y_{1,3} & \ldots & \Delta Y_{1,t} \\ \Delta Y_{2,\pi_2(1)} & \Delta Y_{2,\pi_2(2)} & \Delta Y_{2,\pi_2(3)} & \ldots & \Delta Y_{2,\pi_2(t)} \\ \Delta Y_{3,\pi_3(1)} & \Delta Y_{3,\pi_3(2)} & \Delta Y_{3,\pi_3(3)} & \ldots & \Delta Y_{3,\pi_3(t)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \Delta Y_{n,\pi_n(1)} & \Delta Y_{n,\pi_n(2)} & \Delta Y_{n,\pi_n(3)} & \ldots & \Delta Y_{n,\pi_n(t)} \end{bmatrix}.$$

Define $\tilde{\Sigma} = \frac{1}{t} \Delta \tilde{Y} \cdot \Delta \tilde{Y}^T$ as the null covariance matrix. Repeat this for R times and compute the eigenvalues of $\tilde{\Sigma}_r$ for each $1 \leq r \leq R$. Evaluate the p-value for each estimated eigenvalue $\hat{\lambda}_k$ by $(N_k+1)/(R+1)$ where N_k is the counts that $\hat{\lambda}_k$ is less than the k-th largest eigenvalue of $\tilde{\Sigma}_r$ over $1 \leq r \leq R$. Eigenvalues with small p-values indicate that they are less likely arising from the spectrum of a randomly permuted matrix and thus considered to be signal. Draw your own conclusion with your observations and analysis on this data. A reference is: Buja and Eyuboglu, "Remarks on Parallel Analysis", Multivariate Behavioral Research, 27(4): 509-540, 1992.

3. **Finite rank perturbations of random symmetric matrices:* Wigner's semi-circle law (proved by Eugene Wigner in 1951) concerns the limiting distribution of the eigenvalues of random symmetric matrices. It states, for example, that the limiting eigenvalue distribution of $n \times n$ symmetric matrices whose entries w_{ij} on and above the diagonal ($i \leq j$) are i.i.d Gaussians $\mathcal{N}(0, \frac{1}{4n})$ (and the entries below the diagonal are determined by symmetrization, i.e., $w_{ji} = w_{ij}$) is the semi-circle:

$$p(t) = \frac{2}{\pi}\sqrt{1-t^2}, \quad -1 \leq t \leq 1,$$

where the distribution is supported in the interval $[-1, 1]$.

(a) Confirm Wigner's semi-circle law using MATLAB or R simulations (take, e.g., $n = 400$).

(b) Find the largest eigenvalue of a rank-1 perturbation of a Wigner matrix. That is, find the largest eigenvalue of the matrix

$$W + \lambda_0 u u^T,$$

where W is an $n \times n$ random symmetric matrix as above, and u is some deterministic unit-norm vector. Determine the value of λ_0 for which a phase transition occurs. What is the correlation between the top eigenvector of $W + \lambda_0 uu^T$ and the vector u as a function of λ_0? Use techniques similar to the ones we used in class for analyzing finite rank perturbations of sample covariance matrices.

[Some Hints about homework] For Wigner Matrix $W = [w_{ij}]_{n\times n}, w_{ij} = w_{ji}, w_{ij} \sim N(0, \frac{\sigma}{\sqrt{n}})$, the answer is

$$\begin{aligned} \text{eigenvalue is} \qquad & \lambda = R + \tfrac{1}{R} \\ \text{eigenvector satisfies} \qquad & (u^T \hat{v})^2 = 1 - \tfrac{1}{R^2} \end{aligned}$$