Math 52 - Fall 2010 - Final Exam

Name: ____________________________

Student ID: ________________________

Signature: _________________________

Instructions:

- Print your name and student ID number, select your section number and TA’s name, and write your signature to indicate that you accept the Honor Code.

- There are 10 problems on the pages numbered from 1 to 10, for a total of 100 points. Point values are given in parentheses. Please check that the version of the exam you have is complete, and correctly stapled.

- Read each question carefully. In order to receive full credit, please show all of your work and justify your answers.

- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.

- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.

- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.

Remark: You do not need to simplify your answers, but leave them as numbers, e.g. \(
\left(\frac{1+\sqrt{13}}{3}\right)^6 + \sqrt{6}\) is fine.
Problem 1. (10 pts.) Find the integral:

\[ \int_0^1 \int_{\frac{1}{2}y}^2 \sin(x^2) \, dx \, dy \]

Switch the order of integration.......

\[ \int_0^{2x} \int_{\frac{1}{2}y}^2 \sin(x^2) \, dx \, dy \]

\[ = \int_0^1 \int_0^{2x} \sin(x^2) \, dy \, dx \]

\[ = \int_0^1 2x \sin(x^2) \, dx \]

\[ = -\cos x^2 \bigg|_0^1 \]

\[ = -\cos(1) + 1 \]

\[ = 1 - \cos(1) \]
Problem 2. (10 pts.) Change the order of integration in
\[ \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^0 f(x,y) \, dy \, dx \]

Sketch the region,

\[
\begin{align*}
  y &= -\sqrt{1-(x-1)^2} \\
  y^2 &= 1 - (x-1)^2 \\
  (x-1)^2 + y^2 &= 1 \\
  x &= 1 \pm \sqrt{1-y^2} \\
  y &= -\sqrt{1-(x-1)^2} \\

\end{align*}
\]

Circle of radius 1 centered at (1,0)

\[ \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^0 f(x,y) \, dy \, dx \]

\[ = \int_{-1}^0 \int_{-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} f(x,y) \, dx \, dy \]
Problem 3. (10 pts.) Let $D$ be a region in $\mathbb{R}^2$ whose area is equal to 3 and let
\[ \vec{F}(x, y) = (e^x + y) \hat{i} + (3x - 6y) \hat{j}. \]
Find
\[ \oint_C \vec{F} \cdot d\vec{s} \]
along $C = \partial D$ oriented counterclockwise.

By Green's Theorem,
\[ \oint_C \vec{F} \cdot d\vec{s} = \iint_D \text{curl} \vec{F} \ dA \]
\[ = \iint_D \frac{\partial}{\partial x} (3x - 6y) - \frac{\partial}{\partial y} (e^x + y) \ dA \]
\[ = \iint_D (3 - 1) \ dA \]
\[ = 2 \ \text{Area}(D) \]
\[ = 2 \cdot 3 \]
\[ = 6. \]
Problem 4. (10 pts.) Let \( S \) be the surface cut from the set \( \{ (x, y, z) | z = x^2 - y^2 \} \) by the planes \( x - y = \pm 1 \) and \( x + y = \pm 1 \). Show that

\[
\text{area} (S) = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \sqrt{1 + 2u^2 + 2v^2} \, du \, dv
\]

DO NOT COMPUTE THAT INTEGRAL.

Let \( D \) be the region below:

\[
\begin{align*}
\text{S is the graph of } f(x, y) &= x^2 - y^2 \text{ over } D. \\
\Rightarrow \quad \text{area} (S) &= \iint_D \sqrt{1 + \|\nabla f\|^2} \, dx \, dy \\
&= \iint_D \frac{1}{\sqrt{1 + (2x)^2 + (2y)^2}} \, dx \, dy.
\end{align*}
\]

Do a linear change of coordinate,

\[
\begin{align*}
\begin{cases}
U = x - y \\
V = x + y
\end{cases} \quad \Rightarrow \quad \begin{cases}
x = \frac{1}{2} U + \frac{1}{2} V \\
y = -\frac{1}{2} U + \frac{1}{2} V
\end{cases}
\end{align*}
\]

\[
\frac{\partial (x, y)}{\partial (U, V)} = \left| \text{det} \left( \begin{array}{cc}
\frac{1}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2}
\end{array} \right) \right| = \frac{1}{2}
\]

Hence,

\[
\begin{align*}
\text{area} (S) &= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \sqrt{1 + (U+V)^2 + (U-V)^2} \cdot \frac{1}{2} \, du \, dv \\
&= \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \sqrt{1 + 2U^2 + 2V^2} \, du \, dv.
\end{align*}
\]
Problem 5. (10 pts.) Let \( W \) be the solid in \( \mathbb{R}^3 \) bounded by paraboloid \( z = x^2 + y^2 \) and the plane \( z = y + 2 \). Write the triple iterated integral computing \( \iiint_W f(x, y, z) \, dV \) viewing \( W \) as:

a) as \( z \) simple.

\[
\begin{align*}
W \text{ is defined by:} & \quad \begin{cases} 
2 \geq & x^2 + y^2 \quad \text{(1)} \\
2 \leq & y + 2 \quad \text{(2)} 
\end{cases} \\
\text{Combine (1) and (2),} & \quad x^2 + y^2 \leq y + 2 \\
\Rightarrow & \quad x^2 + y^2 - y + \frac{1}{4} \leq 2 + \frac{1}{4} \\
\Rightarrow & \quad x^2 + (y - \frac{1}{2})^2 \leq \frac{9}{4} \\
\end{align*}
\]

\[
\iiint_W f \, dV = \int_{\frac{3}{2}}^{\frac{9}{4} - x^2} \int_{\frac{1}{2} - \frac{3}{4} - x^2}^{\frac{1}{2} + \frac{1}{2} \sqrt{z - y^2}} \int_{z = y + 2}^{y + 2} f(x, y, z) \, dz \, dy \, dx
\]

b) as \( x \) simple.

\[
\begin{align*}
1 \Rightarrow & \quad x^2 \leq z - y^2 \quad \Rightarrow \quad -z - y^2 \leq x \leq z - y^2 \quad \text{(∗)} \\
\text{and} & \quad z - y^2 \geq 0 \quad \text{(3)} \\
2 \Rightarrow & \quad y^2 \leq z \leq y + 2 \\
\text{(∗∗)} \Rightarrow & \quad y^2 \leq y + 2 \quad \Rightarrow \quad y^2 - y - 2 \leq 0 \quad \Rightarrow \quad (y - 2)(y + 1) \leq 0 \\
\Rightarrow & \quad -1 \leq y \leq 2 \\
\end{align*}
\]

\[
\iiint_W f \, dV = \int_{-1}^{2} \int_{y^2}^{y + 2} \int_{\frac{z - y^2}{y^2}}^{\frac{z - y^2}{y^2}} f(x, y, z) \, dx \, dz \, dy
\]
Problem 6. (10 pts.) Find the integral

\[ \int_{a}^{b} (x^2 + 2xy - y^2) \, dx + (x^2 - 2xy - y^2) \, dy \]

where \( \alpha(t) = (e^{t^2} \sin(\pi t), t^2 - 2t) \) for \( 0 \leq t \leq 1 \).

Let \( \vec{F} = (x^2 + 2xy - y^2, x^2 - 2xy - y^2) \) defined on \( \mathbb{R}^2 \) (simply-connected).

\[
\nabla \times \vec{F} = \frac{\partial}{\partial x} (x^2 - 2xy - y^2) - \frac{\partial}{\partial y} (x^2 + 2xy - y^2) = (2x - 2y) - (2x - 2y) = 0
\]

Therefore, \( \vec{F} \) is conservative (i.e. \( \vec{F} = \nabla f \)).

Solve \( \begin{cases} \frac{df}{dx} = x^2 + 2xy - y^2 \\ \frac{df}{dy} = x^2 - 2xy - y^2 \end{cases} \)  \( \Rightarrow \) \( f = \frac{1}{3} x^3 + x^2 y - y^3 - \frac{1}{3} y^3 + \text{constant} \)

By Fundamental Theorem for Line Integrals,

\[
\int_{\alpha} \vec{F} \cdot d\alpha = f(\alpha(1)) - f(\alpha(0))
\]

\( = f(0, -1) - f(0, 0) \)

\( = \frac{1}{3} - 0 \)

\( = \frac{1}{3} \)
Problem 7. (10 pts.) Let $C$ be the (solid) cylinder $x^2 + y^2 \leq 1$, $1 \leq z \leq 5$ and let $D_+$ be its top (i.e. $x^2 + y^2 \leq 1$, $z = 5$) and $D_-$ be its bottom. Show that

$$\text{vol}(C) = \iiint_{D_+} \vec{F} \cdot d\vec{S}$$

where $\vec{F} = (z - 1) \hat{k}$.

By Gauss' Theorem,

$$\iiint_C \nabla \cdot \vec{F} \, dV = \iint_{\partial C} \vec{F} \cdot \hat{n} \, dS.$$

1. $$\iiint_C \nabla \cdot \vec{F} \, dV = \iiint_C 1 \, dV = \text{Vol}(C)$$

2. $$\iint_{D_-} \vec{F} \cdot \hat{n} \, dS = 0$$ since $\vec{F} = \vec{0}$ on $D_-$ ($z = 1$)

3. On $S$, $\hat{n} = (x, y, 0)$

$$\vec{F} \cdot \hat{n} = (z - 1) \cdot (0, 0, 1) = 0$$

$$\Rightarrow \iint_S \vec{F} \cdot \hat{n} \, dS = 0$$

Therefore,

$$\text{Vol}(C) = \iiint_{D_+} \vec{F} \cdot d\vec{S}$$
Problem 8. (10 pts.) For any surface $S$ let $\vec{n}_S$ denote a unit normal vector to $S$, i.e. a vector perpendicular to $S$ of length one. For this problem you can choose one of the two possible $\vec{n}_S$.
In each case below either give an example of a vector field $\vec{F}$ or show that there is no $\vec{F}$ such that $\nabla \times \vec{F} = \vec{n}_S$.

a) $S$ is the disk $x^2 + z^2 \leq 1$, $y = 3$.

$$\vec{n}_S = (0, 1, 0)$$

Take $\vec{F} = (0, 0, -x)$, then

$$\nabla \times \vec{F} = \begin{vmatrix} \frac{x}{3} & \frac{y}{3} & \frac{z}{3} \\ \frac{y}{3} & \frac{z}{3} & \frac{x}{3} \\ 0 & 0 & -x \end{vmatrix} = (0, 1, 0) = \vec{n}_S$$

b) $S$ is the $z \geq 0$ part of the sphere $x^2 + y^2 + z^2 = 9$.

This is badly formulated problem. For Math 52

NOTE: level problem erase the $z \geq 0$ condition (i.e. consider the whole sphere of radius 3.)
Problem 9. (10 pts.) Let $S$ be the surface of the torus parametrized by

$$
\mathbf{T}(\alpha, \theta) = \begin{bmatrix}
(b + a \sin \alpha) \sin \theta \\
\cos \alpha \\
(b + a \sin \alpha) \cos \theta
\end{bmatrix}
$$

and let $\mathbf{N}_{\alpha, \theta} = \mathbf{T}_\alpha \times \mathbf{T}_\theta$.

a) Find the vector $\mathbf{N}_{\alpha, \theta}$ in terms of $\alpha$ and $\theta$.

$$
\mathbf{T}_\alpha = \frac{\partial \mathbf{T}}{\partial \alpha} = \begin{bmatrix}
(b + a \cos \alpha) \sin \theta \\
-a \sin \alpha \\
(b + a \cos \alpha) \cos \theta
\end{bmatrix}
$$

$$
\mathbf{T}_\theta = \frac{\partial \mathbf{T}}{\partial \theta} = \begin{bmatrix}
(b + a \sin \alpha) \cos \theta \\
0 \\
-(b + a \sin \alpha) \sin \theta
\end{bmatrix}
$$

$$
\mathbf{N}_{\alpha, \theta} = \mathbf{T}_\alpha \times \mathbf{T}_\theta = \begin{bmatrix}
-a \sin \alpha (b + a \sin \alpha) \sin \theta \\
(b + a \cos \alpha) (b + a \sin \alpha) \\
\sin \alpha (b + a \sin \alpha) \cos \theta
\end{bmatrix}
$$

b) Decide if $\mathbf{N}_{\alpha, \theta}$ is pointing in or out of the solid torus bounded by $S$.

At $((b, a, 0), \theta = \frac{\pi}{2}, \alpha = 0)$,

$$
\mathbf{N}_{\alpha, \theta} (0, \frac{\pi}{2}) = (0, b(b+a), 0)
$$

points upward.

Hence, $\mathbf{N}_{\alpha, \theta}$ points out of the solid torus.
Problem 10. (10 pts.) Use Stokes Theorem to compute

\[ \oint_C y \, dx + z \, dy + x \, dz \]

where \( C \) is the circle \( x + y + z = 0, x^2 + y^2 + z^2 = a^2 \) oriented counterclockwise as viewed from above.

Let \( D \) be the disk on \( xy + z = 0 \) bounded by \( C \) oriented by \( \vec{n} = \frac{1}{\sqrt{3}} (1, 1, 1) \).

By Stokes' Theorem, take \( \vec{F} = (y, z, x) \), \( \nabla \times \vec{F} = (-1, -1, -1) \)

\[ \oint_C \vec{F} \cdot d\vec{S} = \iint_D (\nabla \times \vec{F}) \, d\vec{S} \]

\[ = \iint_D (\nabla \times \vec{F}) \cdot \vec{n} \, dS \]

\[ = \iint_D (-1, -1, -1) \cdot \frac{1}{\sqrt{3}} (1, 1, 1) \, dS \]

\[ = \frac{1}{\sqrt{3}} \text{Area}(D) \]

\[ = \frac{1}{\sqrt{3}} \pi a^2 \]

\[ = \frac{\sqrt{3}}{3} \pi a^2 \]
The following boxes are strictly for grading purposes. Please do not mark.

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