Math 52
Second Midterm Solutions 2009

1. (8 points)

Decide whether or not the following vector fields are gradient vector fields. If so, find a potential function for the vector field. If not, explain how you know that no potential function exists.

a) \( F(x, y, z) = (3y + 4xz, 3x - 2y, 2x^2 + 1) \) on all of \( \mathbb{R}^3 \).

**Solution:** First we compute the curl of \( F(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) \). It is equal to \( \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = (0 - 0, 4x - 4x, 3 - 3) = (0, 0, 0) \). Since this equals 0 we know there is a potential function \( \phi(x, y, z) \) on \( \mathbb{R}^3 \).

We must have \( \frac{\partial \phi}{\partial x} = 3y + 4xz \) so that \( \phi(x, y, z) = 3xy + 2x^2z + f(y, z) \). From the partial with respect to \( y \), we obtain the condition that \( 3x - 2y = 3x + \frac{\partial f}{\partial y} \) so \( f(y, z) = y^2 + g(z) \). From the partial with respect to \( z \) we obtain \( 2x^2 + 1 = 2x^2 + \frac{\partial g}{\partial z} \) so that \( g(z) = z + C \) for an arbitrary constant \( C \). Putting this together we conclude that \( \phi(x, y, z) = 3xy + 2x^2z - y^2 + z + C \).

b) \( F(x, y) = (2xe^{-y} + xy^2, x^2e^{-y} + x^2y) \) on all of \( \mathbb{R}^2 \).

**Solution:** We compute the curl, \( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \) of \( F(x, y) = (M(x, y), N(x, y)) \). This equals \( (2x)(e^{-y} + y) - (2xe^{-y} + 2xy) = 4xe^{-y} \). Since this is non-zero \( F(x, y) \), can have no potential function.

2. (6 points)

Let \( F(x, y, z) = (z, 2y, -x) \) be a force field in \( \mathbb{R}^3 \). Calculate the work done by \( F \) on a particle as it moves from \((1, 1, 1)\) to \((8, 4, 2)\) along the curve determined by the conditions \( x = z^3, y = z^2 \).

**Solution:** We parametrize the curve \( C \) as \( \gamma(t) = (t^3, t^2, t) \), where \( 1 \leq t \leq 2 \) and compute that \( \gamma'(t) = (3t^2, 2t, 1) \). The work done equals \( \int_C F \cdot T ds = \int_1^2 F(\gamma(t)) \cdot \gamma'(t) \, dt \). We compute that this equals \( \int_1^2 (t, 2t^2, -t^3) \cdot (3t^2, 2t, 1) \, dt = \int_1^2 6t^3 \, dt = \frac{3}{2}t^4 \bigg|_{t=1}^{t=2} = \left(\frac{3}{2}\right)15 = 45 \).
3. (8 points)
Let $C$ be the piecewise smooth simple closed curve that follows the straight line $y = -x$ from the point $(1, -1)$ to the point $(0, 0)$ and then returns to the point $(1, -1)$ via the parabola $y^2 = x$.

a) Parametrize each of the two parts of the curve $C$. Evaluate $\oint_C (x + y) \, dx + xy \, dy$ using your parametrizations.

**Solution:** We will denote the straight line part of $C$ by $C_1$ and the part on the parabola by $C_2$.

We can parametrize $C_1$ by $\gamma_1(t) = (1-t, -1-t), 0 \leq t \leq 1$ and $C_2$ by $\gamma_2(t) = (t^2, -t), 0 \leq t \leq 1$.

Along these two curves the tangent vectors equal $(-1, 1)$ and $(2t, -1)$ and the vector field $F(x, y)$ equals $(0, -(1-t)^2)$ and $(t^2 - t, -t^3)$, respectively.

The two line integrals $\int_0^1 F(\gamma(t)) \cdot \gamma_i'(t) \, dt$ for $i = 1, 2$ equal $\int_0^1 (0, -(1-t)^2) \cdot (-1, 1) \, dt$ and $\int_0^1 (t^2 - t, -t^3) \cdot (2t, -1) \, dt$. The first equals $\int_0^1 -(1-t)^2 \, dt = \frac{1}{3}(1-t)^3|_0^1 = -\frac{1}{3}$. The second equals $\int_0^1 2t^3 - 2t^2 + 3 \, dt = \int_0^1 3t^3 - 2t^2 \, dt = (\frac{3}{4} t^4 - \frac{2}{3} t^3)|_{t=1} = \frac{3}{4} - \frac{2}{3}$.

Adding these together we obtain $\frac{3}{4} - 1 = -\frac{1}{4}$.

b) Check your result in part a) using Green’s Theorem. Is the vector field $F(x, y) = (x+y, xy)$ conservative? Why or why not?

**Solution:** Green’s Theorem says that for a vector field $F(x, y) = (M(x, y), N(x, y))$ the line integral $\oint_C F(x, y) \cdot T \, ds$ equals the 2-dimensional integral $\iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA$ over the region $D$ whose boundary equals $C$ and $C$ is oriented with $D$ on its left. In this case the region is bounded below by the parabola and above by the straight line and $C$ is oriented with that region on its left. So we can compute the line integral from the 2-dimensional integral $\int_{-1}^{0} \int_{y^2}^{y(y-1)} \, dxdy$. We compute this to be $\int_{-1}^{0} -(y^2 + y)(y-1)dy = \int_{-1}^{0} y - y^3 \, dy = (\frac{1}{2} y^2 - \frac{1}{4} y^4)|_{y=-1}^0 = -\frac{1}{4}$.

This agrees with the answer in a) as it should.

The vector field is NOT conservative since its integral around a closed loop is non-zero. You can arrive at the same conclusion from the fact that its curl equals $y - 1$ which is non-zero.

4. (6 points)
a) Find a potential function for the vector field $G(x, y) = (2xy, x^2 + y^2)$.

**Solution** Denote the potential function by $\phi(x, y)$. From $\frac{\partial \phi}{\partial x} = 2xy$ we know that $\phi(x, y) = x^2 y + f(y)$. From the partial with respect to $y$ we conclude that $x^2 + y^2 = x^2 + \frac{\partial f(y)}{\partial y}$ so that $f(y) = \frac{1}{2} y^3$ plus an arbitrary constant. Thus, $\phi(x, y) = x^2 y + \frac{1}{2} y^3$ is a potential function.

b) For the vector field $G(x, y)$ in part a) compute $\int_C G \cdot ds$, where $C$ is the portion of the curve $\sqrt{x} + xy + \sqrt{y} = 7$ starting at $(4, 1)$ and ending at $(1, 4)$.

**Solution** From the fundamental theorem of line integrals we know that when $G(x, y) = \nabla \phi(x, y)$ we can compute $\int_C G \cdot ds$ by subtracting the value of $\phi$ at the initial point of $C$ from the value at the terminal point. The integral thus equals $\phi(1, 4) - \phi(4, 1) = (4 + \frac{64}{3}) - (16 - \frac{3}{2}) = -12 + 21 = 9.$
5. (8 points)

Find the area of the region in the plane bounded by the $x$-axis and the curve

$$g(t) = (t - \sin t, 1 - \cos t), \ 0 \leq t \leq 2\pi$$

by computing a line integral and applying Green’s Theorem.

**Solution:** Green’s theorem says that for a vector field $F(x, y) = (M(x, y), N(x, y))$ on a region $D$ in the plane that has a closed curve $C$ as its boundary $\oint_C F(x, y) \cdot T \, ds = \iint_D \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \, dA$ when $C$ is oriented with $D$ on its left. For the vector field $F(x, y) = (-y, 0)$ the integrand of the 2-dimensional integral equals 1 so the right hand side of this formula gives the area of the region $D$. Thus we can use the line integral of this vector field around the boundary of $D$ to compute the area. The curve $g(t)$ lies above the $x$-axis, intersecting the axis at its two endpoints. The boundary of the region $D$, oriented with $D$ on its left, is given by moving along the $x$-axis from $x = 0$ to $x = 2\pi$ and then moving backwards along $g(t)$. We will compute the line integral in the opposite direction and then change the sign.

Since $F(x, y) = (0, 0)$ on the $x$-axis there is no contribution from the line integral along that part of the boundary. Along $g(t)$ the tangent vector equals $(1 - \cos t, \sin t)$ so the line integral equals $\int_0^{2\pi} -(1 - \cos t)(1 - \cos t) \, dt = \int_0^{2\pi} -1 + 2\cos t - (\cos t)^2 \, dt$ From the double angle formula we know that $(\cos t)^2 = \left(\frac{1}{2}\right)(1 + \cos 2t)$ so this equals $\int_0^{2\pi} -\frac{3}{2} + 2\cos t - \cos 2t \, dt = -3\pi$ because the integrals of $\cos 2t$ and $\cos t$ from 0 to $2\pi$ equal zero.

Changing the sign we conclude that the area equals $3\pi$.

6. (8 points)

a) Let $B$ denote the unbounded region in the plane defined by $1 \leq x$ and $0 \leq y \leq x$. Decide whether or not the improper integral

$$\int \int_B \frac{y}{x^4} \, dA$$

exists. If it does, calculate it. If it does not exist, explain why not.

**Solution:** We exhaust the region $B$ by the closed, bounded regions $B_n$ where $1 \leq x \leq n$ and $0 \leq y \leq x$. We compute the integral over these regions and take the limit. Because the integrand is positive, if the limit exists, it will equal the value of the improper integral. If the limit does not exist, the sequence of integrals will diverge, and we say that the improper integral does not exist.

On $B_n$ we compute $\int_1^n \int_0^x \frac{y}{x^4} \, dy \, dx = \int_1^n \frac{1}{2} \left(\frac{x^2}{x^4}\right) \, dx = \int_1^n \frac{1}{2x^2} \, dx = -\frac{1}{2x} \big|_{x=1}^{x=n} = \frac{1}{2} - \frac{1}{2n}$. Taking the limit as $n \to +\infty$ we see that the value of the improper integral is $\frac{1}{2}$.

b) The function $f(x, y)$ that equals $4xy \, e^{-(x^2+y^2)}$ in the positive quadrant $x \geq 0$, $y \geq 0$ and equals 0 everywhere else in $\mathbb{R}^2$ is a probability density on $\mathbb{R}^2$ (you DO NOT have to show this; you can assume it’s true). What is the probability that a random point $(x, y) \in \mathbb{R}^2$ satisfies $x \geq 1$ AND $y \geq 1$.

**Hint:** You probably don’t want to change to polar coordinates for this problem.
**Solution:** Since we are given that this function is a probability density; i.e., it is non-negative and its integral over the plane equals 1, the probability is computed by integrating the function over the region of the plane satisfying the given conditions $x \geq 1$ and $y \geq 1$. Since this region is unbounded, this is an improper integral, which we compute integrating over rectangular regions $B_n$ where $1 \leq x \leq n$ and $1 \leq y \leq n$ and taking the limit as $n \to \infty$.

We compute that $\int_1^n \int_1^n 4xy e^{-x^2} e^{-y^2} \, dx \, dy = (-e^{-x^2} |_1^n)(\int_1^n 2y e^{-y^2} \, dy) = (-e^{-x^2} |_1^n)(-e^{-y^2} |_1^n) = (e^{-1} - e^{-n^2})^2$ Taking the limit as $n \to +\infty$ we obtain the value $e^{-2}$ as the probability.

7. (8 points)
Consider the infinite spiral, $C$, parametrized by $(e^{-t} \cos t, e^{-t} \sin t)$, for $0 \leq t < \infty$.

a) Write, as a ratio of integrals with explicit integrands and limits, the average value of the distance to the origin of points on the part of $C$ for $0 \leq t \leq T$. DO NOT EVALUATE.

The average value of a function along a curve equals the scalar integral of that function along the curve divided by the length of the curve:

$$\frac{\int_C f(x, y) \, ds}{\int_C ds}.$$ 

In this case the function is the distance to the origin, which equals $\sqrt{x^2 + y^2}$.

Using the parametrization $\gamma(t) = (e^{-t} \cos t, e^{-t} \sin t)$, we see that $f(\gamma(t))$ equals $e^{-t}$. The tangent vector along the curve equals $(e^{-t}(-\cos t - \sin t), e^{-t}(-\sin t + \cos t))$. This has length squared equal to $e^{-2t}((\cos t)^2 + (\sin t)^2 + 2(\cos t)(\sin t)) + ((\cos t)^2 + (\sin t)^2 - 2(\cos t)(\sin t)) = 2e^{-2t}$. We conclude that $||\gamma'(t)|| = \sqrt{2}e^{-t}$. Therefore the ratio of integrals we would have to compute to find the average value is

$$\frac{\int_0^T f(\gamma(t)) ||\gamma'(t)|| \, dt}{\int_0^T ||\gamma'(t)|| \, dt} = \frac{\int_0^T e^{-t}(\sqrt{2}e^{-t}) \, dt}{\int_0^T \sqrt{2}e^{-t} \, dt}.$$ 

b) Find the limit as $T \to \infty$ of the average value in part a).

Rewriting the integrals in part a) and canceling the two $\sqrt{2}$ terms, we obtain

$$\frac{\int_0^T e^{-2t} \, dt}{\int_0^T e^{-t} \, dt}.$$ 

Evaluating the numerator and denominator gives $-(\frac{1}{2})(e^{-2T} |_0^T)$ and $-(e^{-t} |_0^T)$, respectively. Since $e^{-2T}$ and $e^{-T}$ go to 0 as $T \to +\infty$ the numerator goes to $\frac{1}{2}$ and the denominator goes to 1. So the average distance equals $\frac{1}{2}$.

Note that the calculation of the denominator shows that the infinite spiral has length 1!
8. (8 points)

Consider the region $B$ which is $\mathbb{R}^2$ minus the two points $(-1, 0)$ and $(1, 0)$. Suppose that $F(x, y) = (M(x, y), N(x, y))$ is a vector field defined on all of $B$ and that $M(x, y)$ and $N(x, y)$ have continuous partial derivatives satisfying $\frac{\partial M}{\partial y}(x, y) = -\frac{\partial N}{\partial x}(x, y)$ on all of $B$.

a) Suppose the line integral of $F(x, y)$ around the circle of radius 3 centered at the origin, oriented counter-clockwise, is 5 and the line integral of $F(x, y)$ around the circle of radius 1 around $(-1, 0)$, oriented counter-clockwise, is 3. What is the line integral of $F(x, y)$ around the circle of radius 1 around $(1, 0)$, oriented counter-clockwise? CAREFULLY state the theorem that you are using to find your answer.

Solution: Green’s Theorem states that the integral of curl $F(x, y) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ over a region $D$ equals the line integral of $F(x, y)$ over the boundary of $D$, oriented so that $D$ is on its left. Let $D$ be the region inside the circle of radius 3 centered at the origin and outside the circles of radius 1 centered around the points $(-1, 0)$ and $(1, 0)$, respectively. These three circles form the boundary of $D$; the large circle must be oriented counter-clockwise and the two smaller circles oriented clockwise for $D$ to be on their left.

Since $D$ is a subset of $B$ where the curl of $F(x, y)$ is zero, Green’s Theorem implies that the sum of the line integrals over its boundary circles, oriented as above must equal zero. Let $I_1, I_2, I_3$ denote the respective line integrals around the large circle, the small circle around $(-1, 0)$, and the small circle around $(1, 0)$, all oriented counter-clockwise. Changing orientation changes the sign of a line integral so Green’s Theorem implies that $I_1 - I_2 - I_3 = 0$. We are given that $I_1 = 5$ and that $I_2 = 3$. We conclude that $I_3$, the counter-clockwise integral around the radius 1 circle around $(1, 0)$, equals 2.

b) Draw an oriented closed curve $C$, not necessarily simple (i.e., it might intersect itself), in the region $B$ such that the line integral of $F(x, y)$ around $C$ is 1. Explain your answer.

Solution: Let $C$ be the figure eight curve consisting of the unit circle around $(-1, 0)$, oriented counter-clockwise, and the unit circle around $(1, 0)$, oriented clockwise. The line integral around $C$ equals the sum of the two line integrals around the circles which equals $3 + (-2) = 1$. The negative sign for the loop around $(1, 0)$ comes from the fact that it is oriented clockwise, the opposite direction from the integral computed in part a).