Math 52 - Autumn 2005 - Midterm Exam II

Name: ________________________________

Student ID: __________________________

Signature: ____________________________

Instructions: Print your name and student ID number, write your signature to indicate that you accept the honor code. During the test, you may not use notes, books, calculators. Read each question carefully, and show all your work. There are four problems with the total of 80 points. Point values are given in parentheses. You have 50 minutes to answer all the questions.

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Problem 1. Let \( \vec{F} = \langle -x^2 y, xy^2, \sin x^2 \rangle \) and let \( S \) be part of the paraboloid \( z = 9 - x^2 - y^2 \) above \( xy \)-plane with upper unit normal vector. Find

\[
\iint_S \left( \nabla \times \vec{F} \right) \cdot \vec{n} \, dS
\]

**Hint:** use Stokes theorem.

**Solution:** \( S \) can be parametrized as a graph of \( z = 9 - x^2 - y^2 \) on the disk \( x^2 + y^2 \leq 9 \). We have:

\[
\nabla \times \vec{F} = \langle 0, -2x \cos x^2, x^2 + y^2 \rangle
\]

By Stokes

\[
\iint_S \left( \nabla \times \vec{F} \right) \cdot \vec{n} \, dS = \iint_{D_3} \left( \nabla \times \vec{F} \right) \cdot \vec{n} \, dS
\]

where \( D_3 \) is (flat) disk of radius 3 in \( xy \)-plane, for which \( \vec{n} = \vec{N}(x, y) = \langle 0, 0, 1 \rangle \). Thus:

\[
\iint_S \left( \nabla \times \vec{F} \right) \cdot \vec{n} \, dS = \iint_{D_3} \langle 0, -2x \cos x^2, x^2 + y^2 \rangle \cdot \langle 0, 0, 1 \rangle \, ds =
\]

\[
\int_0^{2\pi} \int_0^3 r^2 \cdot r \, dr \, d\theta = 2\pi \cdot \frac{3^4}{4}
\]

Problem 2. Let \( S \) be a surface cut from half-cylinder \( y^2 + z^2 = 9, z \geq 0 \) by the planes \( x = 0 \) and \( x = 4 \). Find the coordinates of the centroid of \( S \).

**Hint:** Some of the coordinates can be found by the symmetry of \( S \).

**Solution:** By symmetry \( \bar{x} = 2 \) and \( \bar{y} = 0 \). Total mass of \( S \) is equal to the surface area \( \pi \cdot 3 \cdot 4 \). Thus

\[
\bar{z} \cdot 12\pi = \iint_S z \, dS
\]

Parametrize \( S \) with polar \( y - z \) coordinates and \( x = x \), i.e.

\[
\begin{cases}
  x = x \\
  y = 3 \cdot \cos \theta \\
  z = 3 \cdot \sin \theta
\end{cases}
\]

for \( 0 \leq x \leq 4, 0 \leq \theta \leq \pi \), with \( \vec{N}(x, \theta) = \langle 0, -3 \cos \theta, -3 \sin \theta \rangle \), so

\[
\bar{z} \cdot 12\pi = \int_0^4 \int_0^\pi 3 \cdot \sin \theta \cdot 3 \, d\theta \, dx = 72
\]

so: \( \bar{z} = \frac{6}{\pi} \).

Problem 3. Let \( \vec{F} = \langle 2x \cdot \ln y - yz, \frac{x^2}{y} - xz, -xy \rangle \).
a) Show that \( \vec{F} \) is conservative vector field on some region of \( \mathbb{R}^3 \). Describe that region.

**Solution:**

\[
\nabla \times \vec{F} = \begin{bmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2x \cdot \ln y - yz & \frac{x^2}{y} - xz & -xy
\end{bmatrix} = (-x + x, -(y + y), \frac{2x}{y} - z - \frac{2x}{y} + z) = \vec{0}
\]

where \( \vec{F} \) is defined, i.e. on \( y > 0 \)-because of \( \ln y \).

b) Evaluate

\[
\int_{(1,2,1)}^{(2,1,1)} (2x \cdot \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) \, dy - xy \, dz
\]

**Solution:** As both \( y \) coordinates in the above integral are positive, the integral makes sense. We will integrate along the broken path first from \( (2,1,1) \) to \( (1,1,1) \) (along \( x \) coordinate) and then from \( (1,1,1) \) to \( (1,2,1) \) (along \( y \)-coordinate). Get:

\[
= 1 + \ln |y|_1^2 - 1 = \ln 2
\]

**Problem 4.** Find \( \oint_C \vec{F} \cdot ds \) for \( \vec{F} = (e^x \sin y, e^x \cos y) \) and \( C \) the right-hand loop of the graph of the polar equation \( r^2 = 4 \cos \theta \).

**Solution:** By Green’s theorem:

\[
\oint_C \vec{F} \cdot ds = \int_R e^x \sin y \, dx + e^x \cos y \, dy = \iint_R (-e^x \cos y + e^x \cdot \cos y) \, dx \, dy = 0
\]

where \( R \) is the region bounded by the curve \( C \).