• This is a closed-book, closed-notes exam. No calculators or other electronic devices will be permitted. You have 2 hours. If you finish early, you must hand your exam paper to a member of teaching staff.

• In order to receive full credit, please show all of your work and justify your answer. You do not need to simplify your answers unless specifically instructed to do so.

• If you need extra room, use the back sides of each page. There is also a blank page at the end of the exam for you to use. Do not unstaple or detach pages from this exam.

• Please sign the following:

  “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

  Signature: __________________________

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1. (6 points)

Sketch the region of integration \( R \) in the \( xy \)-plane for the integral below. Then reverse the order of integration; i.e., write an iterated integral with explicit limits in which the \( y \)-integration is done first. NO EVALUATION OF INTEGRALS IS REQUIRED.

\[
\int \int _{R} g(x,y) \, dx \, dy = \int _{1}^{5} \int _{0}^{\sqrt{5-y}} g(x,y) \, dx \, dy
\]

**SOLUTION:**

\[
\int _{0}^{2} \int _{1}^{5-x^2} g(x,y) \, dy \, dx
\]
2. (8 points)

Below is an integral over a solid region in $xyz$-space. Rewrite it as an iterated integral in spherical co-ordinates over the same region. Evaluate the integral which is written in spherical co-ordinates.

\[
\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{\sqrt{4-(x^2+y^2)}} \frac{4}{\sqrt{x^2+y^2+z^2}} \, dz \, dy \, dx
\]

**SOLUTION:**

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{1}^{\sqrt{4-(x^2+y^2)}} \frac{4\rho^2 \sin \phi}{\rho} \, d\rho \, d\phi \, d\theta = 4\pi \int_{0}^{\pi} \left( 4 - \frac{1}{\cos^2 \phi} \right) \sin \phi \, d\phi
\]

\[
= 8\pi + 4\pi \int_{1}^{\sqrt{3}} \frac{1}{u^2} \, du \quad (u = \cos \phi)
\]

\[
= 4\pi.
\]
3. (10 points)

Consider the circle which is described in polar coordinates by \( r = 2 \sin \theta \). Let \( D \) be a plate consisting of points inside this circle and outside the circle \( r = 1 \). Suppose the density \( \delta(x, y) \) at a point \((x, y)\) equals \( y \).

a) Set up a double integral in polar co-ordinates, with explicit limits of integration, for the mass of the plate. DO NOT EVALUATE THE INTEGRAL.

**SOLUTION:**

\[
\text{answer} = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2 \sin \theta} r^2 \sin \theta \, dr \, d\theta
\]

Note:

\[
r = 2 \sin \theta \quad \Rightarrow \quad x^2 + y^2 = 2y \quad \Rightarrow \quad x^2 + (y - 1)^2 = 1
\]

b) Set up a formula for the \( y \)-coordinate \( \bar{y} \) of the center of mass of the plate in terms of integrals in polar coordinates with explicit limits of integration. DO NOT COMPUTE the value for \( \bar{y} \). Explain why you know, without computation, that the \( x \)-coordinate for the center of mass is 0.

**SOLUTION:**

Note that both the region and the density function are symmetric across the \( y \)-axis. Consequently \( \bar{x} = 0 \).

\[
\bar{y} = \frac{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2 \sin \theta} r^3 \sin^2 \theta \, dr \, d\theta}{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_1^{2 \sin \theta} r^2 \sin \theta \, dr \, d\theta}
\]
4. (10 points)
Consider the cone in $\mathbb{R}^3$ determined by the condition that the spherical coordinate $\phi$ (angle with positive $z$-axis) satisfies $0 \leq \phi \leq \frac{\pi}{3}$. Let $W$ be the portion of this cone which lies between the sphere of radius 2 and the sphere of radius 1.

a) Compute the volume of $W$.
SOLUTION:

$$\text{volume} = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_1^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$
$$= 2\pi \left(\frac{2^3}{3} - \frac{1^3}{3}\right) \int_0^{\frac{\pi}{3}} \sin \phi \, d\phi$$
$$= \frac{7\pi}{3}$$

b) Find the average distance of a point in $W$ to the origin.
SOLUTION:

$$\text{average distance} = \frac{3}{7\pi} \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_1^2 \rho^3 \sin \phi \, d\rho \, d\phi \, d\theta$$
$$= \frac{3}{7} \left(\frac{2^4}{4} - \frac{1^4}{4}\right)$$
$$= \frac{45}{28}$$
5. (8 points)

Consider the triangular region $D$ in the $(x, z)$-plane bounded by the lines $x = 2$, $z = 1$, and $x = z$. Let $W$ be the 3-dimensional solid obtained by rotating $D$ all the way around the $z$-axis. ($W$ is like a donut with a triangular cross-section.) Set up an iterated integral in cylindrical coordinates that computes the volume of $W$ and then evaluate that integral.

\[
\int_0^{2\pi} \int_1^2 \int_1^r r \, dz \, dr \, d\theta = 2\pi \int_1^2 r^2 - r \, dr
\]

\[
= 2\pi \left( \frac{r^3}{3} - \frac{r^2}{2} \right) \bigg|_1^2
\]

\[
= \frac{5\pi}{3}.
\]
6. (10 points)

Let \( D \) be the diamond-shaped region in the \( xy \) plane with vertices at 
\((0, 0), (\pi, \pi), (0, 2\pi), (-\pi, \pi)\). Consider the transformation \( T \) from the \( uv \) plane to the \( xy \) plane defined by 
\[
T(u, v) = \left( \frac{u-v}{2}, \frac{u+v}{2} \right) = (x, y).
\]

a) Draw a sketch of \( D \) in the \( xy \) plane and find the rectangle \( C \) in the \( uv \) plane which is taken by \( T \) to \( D \).

SOLUTION:

b) Set up an integral (using the transformation \( T \) as a change of coordinates) over the region \( C \) in the \( uv \)-plane that can be used to compute the following integral over \( D \):
\[
\int \int_D (x-y)^2 \sin^2(x+y) \, dx \, dy.
\]

Then COMPUTE the integral over \( C \).

SOLUTION: Observe,
\[
(x - y) = \frac{u-v}{2} - \frac{u+v}{2} = -v
\]
\[
(x + y) = \frac{u-v}{2} + \frac{u+v}{2} = u
\]
\[
\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \left| \begin{array}{cc} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{array} \right| = \frac{1}{2}.
\]

Thus
\[
\int \int_D (x-y)^2 \sin^2(x+y) \, dx \, dy = \int \int_C (-v)^2(\sin^2 u) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \, du \, dv
\]
\[
= \frac{1}{2} \int_0^{2\pi} \int_0^{2\pi} v^2 \sin^2 u \, du \, dv
\]
\[
= \frac{4\pi^3}{3} \int_0^{2\pi} \sin^2(u) \, du = \frac{4\pi^4}{3} \quad \text{use } \sin^2 u = \frac{1 - \cos(2u)}{2}
\]
7. (8 points)

a) Set up an iterated integral that computes the integral of \( f(x, y, z) = x \) over the following region in \( \mathbb{R}^3 \):

\[
0 \leq x, \ 0 \leq y, \ x + y \leq 1, \ 0 \leq z \leq x + y.
\]

Integrate first in the \( z \)-direction and then the other two directions in either order. Do NOT evaluate the integral.

**SOLUTION:**

\[
\int_0^1 \int_0^{1-x} \int_0^{x+y} x \, dz \, dy \, dx
\]

b) Sketch the region in part a). Set up the limits of integration in order to repeat part a), this time integrating first in the \( y \)-direction. Do NOT compute the integral(s).

**SOLUTION:**

\[
\int_0^1 \int_0^x \int_0^{1-x} x \, dy \, dz \, dx + \int_0^1 \int_x^{1-x} \int_0^{1-x} x \, dy \, dz \, dx
\]