Math 52 - Winter 2006 - Midterm Exam I

Problem 1. (8 pts.) Let \( R \) be a region of \( \mathbb{R}^2 \) symmetrical about the \( y \)-axis, and \( R_+ \) be the “half” of the region \( R \) contained in the half plane \( x \geq 0 \). Mark as TRUE/FALSE the following statements:

a) \[ \int\int_R dA = 2 \cdot \int\int_{R_+} dA \] TRUE

b) \[ \int\int_R x \, dA = 2 \cdot \int\int_{R_+} x \, dA \] FALSE (in fact \[ \int\int_R x \, dA = 0. \])

c) \[ \int\int_R \sin(y) \, dA = 2 \cdot \int\int_{R_+} \sin(y) \, dA \] TRUE

d) \[ \int\int_R x^2 \, dA = 2 \cdot \int\int_{R_+} x^2 \, dA \] TRUE

Problem 2. (12 pts.) Evaluate the integral \( \iiint_T x^2 \, dV \), where \( T \) is the tetrahedron bounded by coordinate planes and the plane \( 2x + 3y + z = 6 \).

Solution:

\[
\int\int\int_T x^2 \, dV = \int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} x^2 \, dz \, dy \, dx = \int_0^3 \int_0^{2-\frac{2}{3}x} x^2 (6 - 2x - 3y) \, dy \, dx =
\]

\[
= \int_0^3 x^2 (6 - 2x) \left( 2 - \frac{2}{3}x \right) - \frac{3}{2} x^2 \left( 2 - \frac{2}{3}x \right)^2 \, dx = \int_0^3 x^2 \left( 6 - 4x + \frac{2}{3} x^2 \right) \, dx = \frac{27}{5}
\]

Problem 3. (12 pts.) Setup, but do not evaluate the triple integral representing the volume of the bounded solid \( W \) bounded by the graphs of the cylinders \( x^2 + z^2 = 1 \), \( y^2 + z^2 = 1 \) that is contained the first octant of the coordinate system.

Solution: First octant implies \( x \geq 0 \), \( y \geq 0 \) and \( z \geq 0 \). We consider \( W \) as \( z \)-simple. Then the projection is the square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \), and the bounds for \( z \) are:

\[
0 \leq z \leq \min \left( \sqrt{1 - x^2}, \sqrt{1 - y^2} \right)
\]
thus the upper bound is $\sqrt{1-x^2}$ if

$$\sqrt{1-x^2} \leq \sqrt{1-y^2}$$
if $1-x^2 \leq 1-y^2$

if $y^2 \leq x^2$

which for $x \geq 0$, $y \geq 0$ means $y \leq x$. Thus:

$$V = \int_0^1 \int_0^x \int_0^{\sqrt{1-x^2}} dz \ dy \ dx + \int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} dz \ dx \ dy$$

Note: because of symmetries you could take twice of any one of the above integrals.

**Problem 4.** (12 pts.) Evaluate

$$\int_1^e \int_{\ln y}^1 \frac{1}{y} \cdot e^{x^2} \ dx \ dy$$

**Solution:**

$$\int_1^e \int_{\ln y}^1 \frac{1}{y} \cdot e^{x^2} \ dx \ dy = \int_0^1 \int_0^{e^x} \frac{1}{y} \cdot e^{x^2} \ dy \ dx = \int_0^1 e^{x^2} \ln y \bigg|_1^{e^x} \ dx = \int_0^1 e^{x^2} x \ dx = \frac{1}{2} (e - 1)$$

**Problem 5.** (12 pts.) Change the order of integration in

$$\int_{-1}^2 \int_0^{\vert x \vert} \sin(x + 3y) \ dy \ dx$$

DO NOT EVALUATE THE INTEGRAL

**Solution:**

$$\int_0^1 \int_{-y}^{-y} \sin(x + 3y) \ dx \ dy + \int_0^2 \int_y^2 \sin(x + 3y) \ dx \ dy$$

**Problem 6.**

1. (4 pts.) Sketch the region in the $xy$–plane described by the inequality

$$\vert x \vert + \vert y \vert \leq 1$$

2. (6 pts.) Show that

$$\int \int_{\vert x \vert + \vert y \vert \leq 1} f(x + y) \ dx \ dy = \int_{-1}^1 f(t) \ dt,$$

for any continuous real-valued function $f$.

**Hint:** Perform some change of variables...
Solution: Perform the change of variables \( u = x + y, \ v = x - y. \) Then
\[
\frac{\partial(u,v)}{\partial(x,y)} = 2
\]
thus:
\[
\int \int_{|x|+|y|\leq 1} f(x+y) \ dx \ dy = \int_{-1}^{1} \int_{-1}^{1} f(u) \cdot \frac{1}{2} \ dv \ du = \int_{-1}^{1} f(u) \ du
\]

Problem 7. (10 pts.) Consider the change of variables \( u = xy, \ v = yz, \ w = xz. \)

a) Find the Jacobian \( \frac{\partial(x,y,z)}{\partial(u,v,w)}. \)

Solution:
\[
\frac{\partial(x,y,z)}{\partial(u,v,w)} = \text{det} \begin{vmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{vmatrix} = 2xyz
\]

b) Find the volume of the region in the first octant enclosed by the hyperbolic cylinders \( xy = 1, \ xy = 4, \ xz = 1, \ xz = 4, \ yz = 4, \ yz = 9. \)

Hint: Use the fact that \( uvw = x^2y^2z^2. \)

Solution: With the change of variables described in a):
\[
V = \int_{1}^{2} \int_{1}^{4} \int_{1}^{9} \frac{1}{uvw} \ dwdvdu = 2\sqrt{u} \cdot 2\sqrt{v} \cdot 2\sqrt{w} \left. \right|_{1}^{4} = 72
\]

Problem 8. a) (8 pts.) Set up an integral in cylindrical coordinates that represents
the volume of the region in \( \mathbb{R}^3 \) bounded by the two surfaces \( z = x^2 \) and \( z = 9 - y^2. \)

Solution: The inequalities representing the region are \( x^2 \leq z \leq 9 - y^2. \) Thus the projection
onto \( x - y \) plane is the region \( x^2 + y^2 \leq 9, \) i.e. the disk of radius 9 centered at the origin.

Hence the volume is:
\[
\int_{0}^{2\pi} \int_{0}^{3} \int_{x^2}^{9-y^2} r \ dz \ dr \ d\theta = 2\pi \int_{0}^{3} (9 - r^2) \ r \ dr = \frac{81\pi}{2}
\]

b) (4 pts.) evaluate the integral in part a).

Problem 9. (12 pts.) Let \( S \) be the part of the ball \( x^2 + y^2 + z^2 \leq 4 \) that lies above the
cone \( z = \sqrt{x^2 + y^2}. \) Find the volume of \( S. \)

Hint: use spherical coordinates.

Solution:
\[
V = \int_{0}^{2\pi} \int_{0}^{\pi/4} \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta = 2\pi \cdot \frac{8}{3} \cdot (-\cos \phi) \left. \right|_{0}^{\pi/4} = \frac{16\pi}{3} \left( 1 - \frac{\sqrt{2}}{2} \right)
\]