MATH 52 MIDTERM I OCTOBER 14, 2009

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM. NO CALCULATORS OR OTHER ELECTRONIC DEVICES ARE PERMITTED.

THERE IS ONLY ONE INTEGRAL EVALUATION, IN PROBLEM 3(a). THERE ARE 5 PROBLEMS WORTH 10 POINTS EACH, ON 7 PAGES (INCLUDING COVER PAGE).

Please sign the following, and print your name neatly:

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

SIGNATURE: ____________________________

Here are some formulas relating rectangular coordinates $x, y, z$, cylindrical coordinates $r, \theta, z$, and spherical coordinates $\rho, \phi, \theta$:

\[
\begin{align*}
    z &= \rho \cos(\phi) \\
    r &= \rho \sin(\phi) \\
    x &= r \cos(\theta) = \rho \sin(\phi) \cos(\theta) \\
    y &= r \sin(\theta) = \rho \sin(\phi) \sin(\theta).
\end{align*}
\]

<table>
<thead>
<tr>
<th>Problem</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
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<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>
1. Suppose $D$ is a thin plate in the first quadrant bounded by the $y$-axis, the curve $y = x^2$ and the line $x + y = 2$. Suppose the mass density function on $D$ is given by

$$
\delta(x, y) = \begin{cases} 
x & \text{if } 0 \leq y \leq 1 \\
xy & \text{if } 1 \leq y
\end{cases}
$$

(a) Sketch plate $D$.

The line $x + y = 2$ connects $(2, 0)$ and $(0, 2)$, and intersects the parabola $y = x^2$ at $(1, 1)$ in the first quadrant. So the shape is triangle-like, with straight edges from $(0, 0)$ to $(0, 2)$ and from $(0, 2)$ to $(1, 1)$, then a curved edge on the parabola from $(0, 0)$ to $(1, 1)$.

(b) Express the total mass of plate $D$ in terms of one or more explicit iterated integrals in the order $\int \int - dy \ dx$. DO NOT EVALUATE.

[Explicit means specific integrands and limits of integration, not symbols like $\delta$.]

Although the shape has an easy $y$-simple description $0 \leq x \leq 1, x^2 \leq y \leq 2 - x$, since the density changes formulas across the line $y = 1$ in order to get explicit integrals it is necessary to regard the plate as two separate $y$-simple regions and add the masses of each, in order to write $\int_0^1 \int_{x^2}^{2-x} \delta(x, y) \ dy \ dx$ in terms of explicit integrals.

Total mass = $\int_0^1 \int_{x^2}^{2-x} x \ dy \ dx + \int_1^2 \int_1^{2-y} xy \ dy \ dx$.

(c) Express the total first moment of plate $D$ with respect to the $y$-axis ($x = 0$), in terms of one or more explicit integrated integrals in the order $\int \int - dx \ dy$. DO NOT EVALUATE.

Regarded as an $x$-simple region, the "upper" boundary curve $x = d(y)$ is now given by two separate formulas, $x = d(y) = \sqrt{y}$ for $0 \leq y \leq 1$ and $x = d(y) = 2 - y$ for $1 \leq y \leq 2$. The moment with respect to the line $x = 0$ is the integral $\int \int_D x\delta(x, y) \ dx \ dy$. Again we must also take into account the two formulas for $\delta$, as well as the two upper boundary curves, in order to get explicit integrals.

Total moment $M_{(x=0)} = \int_0^1 \int_0^{\sqrt{y}} x^2 \ dx \ dy + \int_1^2 \int_0^{2-y} x^2 \ dy \ dx$.
2. Consider a solid \( W \) in the first octant \( x \geq 0, \ y \geq 0, \ z \geq 0 \) that is also bounded by the two cylinders \( x^2 + y^2 = 4 \) and \( y^2 + z^2 = 4 \). Suppose the electrical charge density on \( W \) is given by the function \( x \).

(a) Express the total electrical charge on \( W \) as an explicit iterated integral in rectangular coordinates in the order \( \int \int \int -dz \ dx \ dy \). DO NOT EVALUATE.

The order of integration demands that we view the solid as \( z \)-simple. It is in the first quadrant, and the shadow in the \( xy \)-plane will be the quarter disk \( 0 \leq x, \ 0 \leq y, \ x^2 + y^2 \leq 4 \) because the solid is inside the cylinder \( x^2 + y^2 = 4 \). In rectangular coordinates the shadow is then \( 0 \leq y \leq 2, \ 0 \leq x \leq \sqrt{4 - y^2} \). The ‘floor’ of the solid in the \( z \)-direction is \( z = 0 \) and the ‘ceiling’ is \( z = \sqrt{4 - y^2} \), since the solid is also inside the cylinder \( y^2 + z^2 = 4 \). So the answer is
\[
\int_0^2 \int_0^\sqrt{4-y^2} \int_0^{\sqrt{4-y^2}} x \ dz \ dx \ dy.
\]

(b) Express the total electric charge on \( W \) as an explicit integral in cylindrical coordinates. DO NOT EVALUATE.

The shadow is now described in polar coordinates by \( 0 \leq \theta \leq \pi/2, \ 0 \leq r \leq 2 \). We then just change the integral in part (a) to cylindrical coordinates, and get
\[
\int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{4-r^2\sin^2\theta}} r\cos\theta \ r \ dz \ dr \ d\theta.
\]
3(a). EVALUATE the following double integral:
\[ \int \int_R e^{(-x^2-y^2)} \, dA, \]
where \( R \) is the plane region defined by \( 1 \leq x^2 + y^2 \leq 2 \).
We use polar coordinates
\[ \int_0^{2\pi} \int_1^{\sqrt{2}} e^{-r^2} r \, dr \, d\theta = 2\pi \left( \frac{1}{e} - \frac{1}{e^2} \right). \]

(b) Consider a spatial object \( W \) inside the sphere \( x^2 + y^2 + z^2 = 4 \) and above the cone \( z = r = \sqrt{x^2 + y^2} \), with density given by \( \delta(x, y, z) = z \). If \((\bar{x}, \bar{y}, \bar{z})\) is the center of mass of \( W \), then two of the three coordinates are 0. Which two, and briefly say why? Write a formula for the third center of mass coordinate of as a quotient of two explicit integrals expressed in spherical coordinates. DO NOT EVALUATE.

The object is an ‘ice cream cone’. The surfaces \( x^2 + y^2 + z^2 = 4 \) and \( z = r = \sqrt{x^2 + y^2} \) intersect over the circle \( x^2 + y^2 = 2 \) and in the plane \( z = \sqrt{2} \). Thus the spherical coordinate description of the object is \( 0 \leq \phi \leq \pi/4 \) \( 0 \leq \rho \leq 2 \), \( 0 \leq \theta \leq 2\pi \). The object is symmetric about the \( xz \)-plane and the \( yz \)-plane. Also, the density function \( z \) is symmetric about these planes, so each piece of mass-moment on one side of these planes is balanced by an equivalent mass-moment piece on the other side. So \( \bar{y} = 0 \) and \( \bar{x} = 0 \).

\[ \bar{z} = \frac{M_{(z=0)}}{\text{total mass}} = \frac{\int \int_W z \delta \, dV}{\int \int_W \delta \, dV} = \frac{\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 (\rho \cos(\phi))^2 \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta}{\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho \cos(\phi) \rho^2 \sin(\phi) \, d\rho \, d\phi \, d\theta}. \]
4(a). Consider a thin plate in the $xy$-plane over a region $R$, with mass density given by $\delta(x, y) = e^x e^y$ for $(x, y) \in R$. Write down a formula as an integral over region $R$ for the moment $M_{(y=b)}$ of plate $R$ with respect to the horizontal line $y = b$, where $b$ is some constant.

$$M_{(y=b)} = \int \int_R (y - b)e^x e^y \, dxdy$$

(b) Derive the value of $b$ from part (a) so that $M_{(y=b)} = 0$. Your answer should be expressed in terms of integrals over region $R$.

Setting $0 = \int \int_R (y - b)e^x e^y \, dxdy$, we get

$$0 = \int \int_R ye^x e^y \, dxdy - b \int \int_R e^x e^y \, dxdy = \int \int_R ye^x e^y \, dxdy - b \int \int_R e^x e^y \, dxdy.$$

We solve for $b$ and get

$$b = \frac{\int \int_R ye^x e^y \, dxdy}{\int \int_R e^x e^y \, dxdy}.$$

(c) If the shape of plate $R$ in part (a) is symmetrical about the $y$-axis, what can you say about the center of mass $(\bar{x}, \bar{y})$ of the plate?

Always $\bar{y} = b$, where $b$ is the value calculated in part (b). With the assumed symmetry about the $y$-axis, you can also say $\bar{x} > 0$ since geometrically small pieces of the plate near $(x, y)$ for $x > 0$ match small pieces near $(-x, y)$, but the density $e^x e^y > e^{-x} e^y$ is greater on the positive side. So there is more moment on the positive side.
5(a). Suppose $R^*$ is the rectangle $a \leq u \leq b$, $c \leq v \leq d$ in the $uv$-plane. Partition $R^*$ into small rectangles using a grid of vertical and horizontal lines at $u$-values $a = u_0 < u_1 < u_2 < \ldots < u_n = b$ and $v$-values $c = v_0 < v_1 < v_2 < \ldots < v_m = d$. Suppose $T(u,v) = (x(u,v), y(u,v))$ is a 1-1 differentiable transformation from the $uv$-plane to the $xy$-plane, with $T(R^*) = R$, a region in the $xy$-plane. If $f(x,y)$ is a continuous function on $R$, approximate $\int \int_{R^*} f(x,y) \, dA$ by a double sum involving the points $(u_i, v_j)$, the function $f$, the functions $x(u,v)$ and $y(u,v)$ and their partial derivatives, and the small rectangle widths and heights $\Delta u_i = u_i - u_{i-1}$ and $\Delta v_j = v_j - v_{j-1}$, for $1 \leq i \leq n$ and $1 \leq j \leq m$.

We just want the double sum called for here, no discussion. Some discussion is requested in Problem 5(c).

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(x(u_i, v_j), y(u_i, v_j)) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u}(u_i, v_j) & \frac{\partial x}{\partial v}(u_i, v_j) \\ \frac{\partial y}{\partial u}(u_i, v_j) & \frac{\partial y}{\partial v}(u_i, v_j) \end{pmatrix} \right| \Delta u_i \Delta v_j$$

5(b) Express the limit of your double sum in Problem 5(a), as all $\Delta u_i$ and $\Delta v_j$ approach 0, as an integral over rectangle $R^*$ in the $uv$-plane.

$$\int_{a}^{b} \int_{c}^{d} f(x(u,v), y(u,v)) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \, du \, dv$$
5(c) Give a brief geometric explanation of why your double sum in Problem 5(a) should closely approximate \( \int \int_R f(x, y) \, dA \) if all \( \Delta u_i \) and \( \Delta v_j \) are very small.

Nobody said transformation \( T \) was linear.

The \( mn \) small rectangles in \( R^* \) whose sides are vectors \((\Delta u_i, 0)\) and \((0, \Delta v_j)\) are transformed by \( T \) into \( mn \) small regions \( A_{ij} \) in \( R \) with perhaps 4 curved sides. These small regions fill up \( R \), and their diameters will be small if all \( \Delta u_i \) and \( \Delta v_j \) are very small.

Let \( \Delta A_{ij} \) denote the area of small piece \( A_{ij} \). The point \((x_i, y_j) = T(u_i, v_j) = (x(u_i, v_j), y(u_i, v_j))\) lies in small piece \( A_{ij} \) in \( R \). Therefore the double integral \( \int \int_R f(x, y) \, dA \) will be closely approximated by the Riemann sum

\[
\sum_{i=1}^n \sum_{j=1}^m f(x(u_i, v_j), y(u_i, v_j)) \Delta A_{ij}
\]

since the definition of the double integral \( \int \int_R f(x, y) \, dA \) is the limit of such double sums as the diameters of the pieces goes to 0.

But piece \( A_{ij} \) is almost a parallelogram, obtained by replacing \( T \) by its best ‘linear approximation’ at point \((u_i, v_j)\). The sides of this approximating parallelogram are determined by the derivative of \( T \) at point \((u_i, v_j)\). Specifically, the sides of this small parallelogram are the vectors \( \Delta u_i (\partial x/\partial u(u_i, v_j), \partial y/\partial u(u_i, v_j)) \) and \( \Delta v_j (\partial x/\partial v(u_i, v_j), \partial y/\partial v(u_i, v_j)) \).

So the areas \( \Delta A_{ij} \) are, with very small percentage area, approximated by

\[
\left| \det \begin{pmatrix} \partial x/\partial u(u_i, v_j) & \partial x/\partial v(u_i, v_j) \\ \partial y/\partial u(u_i, v_j) & \partial y/\partial v(u_i, v_j) \end{pmatrix} \right| \Delta u_i \Delta v_j.
\]

So you can replace the terms \( \Delta A_{ij} \) in the double sum above by these last determinant expressions, and the limit will be the same, namely \( \int \int_R f(x, y) \, dA \). But this substitution gives exactly the double sum given as the answer to part (a) of this question, namely

\[
\sum_{i=1}^n \sum_{j=1}^m f(x(u_i, v_j), y(u_i, v_j)) \left| \det \begin{pmatrix} \partial x/\partial u(u_i, v_j) & \partial x/\partial v(u_i, v_j) \\ \partial y/\partial u(u_i, v_j) & \partial y/\partial v(u_i, v_j) \end{pmatrix} \right| \Delta u_i \Delta v_j.
\]