• This is a closed-book, closed-notes exam. No calculators or other electronic devices will be permitted. **You have 2 hours.** If you finish early, you must hand your exam paper to a member of teaching staff.

• In order to receive full credit, please show all of your work and justify your answer. You do not need to simplify your answers unless specifically instructed to do so.

• If you need extra room, use the back sides of each page. There is also a blank page at the end of the exam for you to use. Do not unstaple or detach pages from this exam.

• Please sign the following:

  “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

  Signature: ______________________________

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1. (8 points)

Decide whether or not the following vector fields are gradient vector fields. If so, find a potential function for the vector field. If not, explain how you know that no potential function exists.

a) \( F(x, y, z) = (3y + 4xz, 3x - 2y, 2x^2 + 1) \) on all of \( \mathbb{R}^3 \).

b) \( F(x, y) = (2xe^{-y} + xy^2, x^2e^{-y} + x^2y) \) on all of \( \mathbb{R}^2 \).
2. (6 points)

Let $F(x, y, z) = (z, 2y, -x)$ be a force field in $\mathbb{R}^3$. Calculate the work done by $F$ on a particle as it moves from $(1, 1, 1)$ to $(8, 4, 2)$ along the curve determined by the conditions $x = z^3$, $y = z^2$. 
3. (8 points)

Let $C$ be the piecewise smooth simple closed curve that follows the straight line $y = -x$ from the point $(1, -1)$ to the point $(0, 0)$ and then returns to the point $(1, -1)$ via the parabola $y^2 = x$.

a) Parametrize each of the two parts of the curve $C$. Evaluate $\oint_C (x + y) \, dx + xy \, dy$ using your parametrizations.

b) Check your result in part a) using Green’s Theorem. Is the vector field $F(x, y) = (x + y, xy)$ conservative? Why or why not?
4. (6 points)
   a) Find a potential function for the vector field \( G(x, y) = (2xy, x^2 + y^2) \).

b) For the vector field \( G(x, y) \) in part a) compute \( \int_C G \cdot ds \), where \( C \) is the portion of the curve \( \sqrt{x} + xy + \sqrt{y} = 7 \) starting at \((4,1)\) and ending at \((1,4)\).
5. (8 points)

Find the area of the region in the plane bounded by the \( x \)-axis and the curve

\[ g(t) = (t - \sin t, 1 - \cos t), \ 0 \leq t \leq 2\pi \]

by computing a line integral and applying Green’s Theorem.
6. (8 points)
   a) Let $B$ denote the unbounded region in the plane defined by $1 \leq x$ and $0 \leq y \leq x$. Decide whether or not the improper integral
   $$\int \int_B \frac{y}{x^4} \, dA$$
   exists. If it does, calculate it. If it does not exist, explain why not.

   b) The function $f(x, y)$ that equals $4xy e^{-(x^2+y^2)}$ in the positive quadrant $x \geq 0$, $y \geq 0$ and equals 0 everywhere else in $\mathbb{R}^2$ is a probability density on $\mathbb{R}^2$ (you DO NOT have to show this; you can assume it’s true). What is the probability that a random point $(x, y) \in \mathbb{R}^2$ satisfies $x \geq 1$ AND $y \geq 1$.

   Hint: You probably don’t want to change to polar coordinates for this problem.
7. (8 points)
Consider the infinite spiral, $C$, parametrized by $(e^{-t}\cos t, e^{-t}\sin t)$, for $0 \leq t < \infty$.

a) Write, as a ratio of integrals with explicit integrands and limits, the average value of the distance to the origin of points on the part of $C$ for $0 \leq t \leq T$. DO NOT EVALUATE.

b) Find the limit as $T \to \infty$ of the average value in part a).
8. (8 points)

Consider the region $B$ which is $\mathbb{R}^2$ minus the two points $(-1, 0)$ and $(1, 0)$. Suppose that $F(x, y) = (M(x, y), N(x, y))$ is a vector field defined on all of $B$ and that $M(x, y)$ and $N(x, y)$ have continuous partial derivatives satisfying $\frac{\partial M}{\partial y}(x, y) = \frac{\partial N}{\partial x}(x, y)$ on all of $B$.

a) Suppose the line integral of $F(x, y)$ around the circle of radius 3 centered at the origin, oriented counter-clockwise, is 5 and the line integral of $F(x, y)$ around the circle of radius 1 around $(-1, 0)$, oriented counter-clockwise, is 3. What is the line integral of $F(x, y)$ around the circle of radius 1 around $(1, 0)$, oriented counter-clockwise? CAREFULLY state the theorem that you are using to find your answer.

b) Draw an oriented closed curve $C$, not necessarily simple (i.e., it might intersect itself), in the region $B$ such that the line integral of $F(x, y)$ around $C$ is 1. Explain your answer.
Scratch Paper