MATH 52 MIDTERM I APRIL 22, 2009

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM. NO CALCULATORS OR OTHER ELECTRONIC DEVICES ARE PERMITTED.

YOU DO NOT NEED TO EVALUATE ANY INTEGRALS IN ANY PROBLEM. THERE ARE 4 PROBLEMS, EACH WORTH 10 POINTS.

Please sign the following:

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

SIGNATURE: ____________________________

Here are some formulas relating rectangular coordinates \(x, y, z\), cylindrical coordinates \(r, \theta, z\), and spherical coordinates \(\rho, \phi, \theta\):

\[
\begin{align*}
  z &= \rho \cos(\phi) \\
  r &= \rho \sin(\phi) \\
  x &= r \cos(\theta) = \rho \sin(\phi) \cos(\theta) \\
  y &= r \sin(\theta) = \rho \sin(\phi) \sin(\theta).
\end{align*}
\]

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1. Let $D$ be the bounded plane region in the first quadrant bounded by the $x$-axis, the curve $y = x^2$ and the line $x + y = 2$. If $f$ is a continuous function on $D$, express $\int \int_D f \, dA$ in terms of iterated integrals in both orders $\int \int dx \, dy$ and $\int \int dy \, dx$. 

2. Consider a thin plate regarded as the plane region described by \(0 \leq x, \ 1 \leq x^2 + y^2 \leq 4\), with density \(\delta(x, y) = x\).

(a) Sketch the region, and describe the region as a rectangle in polar coordinates \((r, \theta)\)

(b) One of the coordinates \((\bar{x}, \bar{y})\) of the center of mass is 0. State which one, and give a very brief explanation.

(c) Write down a formula for the non-zero center of mass coordinate, as a quotient of two integrals each expressed in polar coordinates.
3. Consider a spatial object in the first octant $0 \leq x, \ 0 \leq y, \ 0 \leq z$ inside the sphere $x^2 + y^2 + z^2 = 1$ and with density $\delta(x, y, z) = z$.

(a) Set up an iterated integral in rectangular coordinates in the order $\int \int \int dz \ dy \ dx$ for the mass of the object.

(b) Set up an integral in spherical coordinates for the moment with respect to $x = 0$ (the $yz$-plane) of the same object.
4. Write T (true) or F (false) in the margin to the left of each statement below. Score is Right minus \( \frac{\text{Wrong}}{2} \), with a minimum of 0. There are ten statements.

(i) If \( f(x) \) and \( g(y) \) are any two continuous functions of one variable then the double integral of the product \( f(x)g(y) \) over the rectangle \( a \leq x \leq b, \ c \leq y \leq d \) can always be computed as the product \( (\int_{a}^{b} f(x)dx)(\int_{c}^{d} g(y)dy) \).

(ii) If \( F(x, y) \) is a function of two variables, then the definition of the double integral of \( F \) over the rectangle \( a \leq x \leq b, \ c \leq y \leq d \) is the iterated integral \( \int_{c}^{d} (\int_{a}^{b} F(x, y)dx) \ dy \).

(iii) If \( \delta(x, y, z) > 0 \) is continuous and gives the density of a solid object occupying region \( W \) in space, then the triple integral of \( \delta(x, y, z) \) over the region \( W \) gives the mass of the object.

(iv) \( \int_{1}^{0} \int_{y}^{2} f(x, y)dx \ dy = \int_{0}^{1} \int_{\sqrt{x}}^{x} f(x, y)dy \ dx \), for all continuous functions \( f(x, y) \).

(v) \( 0 < \int \int_{D} \cos^{2}(x^{3} + y^{3})dA < \pi^{2} \), where \( D \) is the square \( 0 \leq x \leq \pi, \ 0 \leq y \leq \pi \).
(vi) The region in the plane that lies above the graph \( y = x^2 \) and below the graph \( y = 2x \) is both a type I and a type II region in the plane.

(vii) If \( r(\theta) > 0 \) is a continuous, positive function defined for \(-\pi \leq \theta \leq \pi\), then the area in the right half of the xy-plane between the y-axis and the curve described in polar coordinates by \( r = r(\theta) \) is given by \( \int_{-\pi/2}^{\pi/2} \int_{0}^{r(\theta)} 1 \, dr \, d\theta \).

(viii) If a bounded object in three dimensional space \( \mathbb{R}^3 \) has constant density, then the x-coordinate of the center of mass of the object is the average value of the function \( f = x \) on the object.

(ix) If \( W \) is the bounded region of space inside the sphere \( x^2 + y^2 + z^2 = 4 \) and above the plane \( z = 1 \), then there is a rectangular box \( W^* \) in \( (\rho, \phi, \theta) \) space that maps to \( W \) under the standard spherical coordinate transformation \( (x, y, z) = S(\rho, \phi, \theta) \).

(x) If \( T(u, v, w) = (x, y, z) \) is a linear transformation, then for any region \( W^* \) in \( uvw \)-space that has a volume, the volume of the transformed region \( W = T(W^*) \) in \( xyz \)-space is given by \( \text{vol}(W) = |\text{det}(T)| \text{vol}(W^*) \).