Math 52 - Winter 2013 - Final Exam

Name: ____________________________________________

Student ID: ______________________________________

Select your section:

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<tr>
<th>Alexandr Zamorzaev</th>
<th>Junsoo Ha</th>
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Signature: ________________________________________

Instructions:

• Print your name and student ID number, select your section number and TA’s name, and write your signature to indicate that you accept the Honor Code.

• There are 11 problems on the pages numbered from 1 to 11, for a total of 110 points. Point values are given in parentheses. Please check that the version of the exam you have is complete, and correctly stapled.

• Read each question carefully. In order to receive full credit, please show all of your work and justify your answers.

• You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.

• **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.

• If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
Problem 1. (10 pts.) Sketch the region of integration and then change the order of the integration in:

\[ \int_{1/2}^{1} \int_{1-x}^{1-x^2} f(x, y) \, dy \, dx \]
Problem 2. (10 pts.) Let $S$ be the surface described by:

\[
\begin{cases}
(x + 1)^2 + y^2 = 1 \\
y > 0 \\
0 \leq z \leq 4
\end{cases}
\]

Thus $S$ is part of a cylinder.

Let $S$ be oriented with $\mathbf{n}_S \circ \mathbf{j} > 0$.

Give an orientation preserving parametrization of $S$.

Provide formulas and limits for your parameters as well as the order in which the parameters should be used.
Problem 3. (10 pts.) Let $S_R$ be the sphere centered at $(3, 4, 0)$ and radius $R$. Below $\vec{r} = [x, y, z]$ and $r = \| \vec{r} \|$.

a) For radius $R = 1$ use the divergence theorem to show that the outer flux of $\vec{F} = \frac{1}{r^3} \vec{r}$ through $S_1$ is zero.

b) Compute the outer flux of $\vec{F} = \frac{1}{r^3} \vec{r}$ through $S_{10}$ (i.e. the sphere centered at $(3, 4, 0)$ of radius 10).

Hint: first compute the flux of $\vec{F}$ through the sphere $\rho = 1$, i.e. the radius 1 sphere centered at the origin. Then relate that integral to the one you need to compute. Carefully explain which orientations you are using.
Problem 4. (10 pts.)
Find the integral
\[ \int_C (2x + y) \, dx + x^2 \, dy \]
over the closed curve \( C \) shown on the picture on the right.

(b) Let \( \vec{F} = [(x + y), x^2] \). Use the Green theorem to compute the integral
\[ \int_C \vec{F} \cdot \vec{n}_{\text{out}} \, ds \]
for the same curve \( C \).
Problem 5. (10 pts.)
a) Let $D$ be the disk radius 4 centered at $(0, 4)$ with uniform density $\rho \equiv 1$. Find the moment of inertia $I_{(0,0)}$ of $D$ when rotated about the origin $(0, 0)$.

b) Use the theorem about moment of inertia about two parallel axis and the result of the previous part to find the moment of inertia $I_{(3,0)}$ for the disk $D$ described in part (a).
Problem 6. (10 pts.) Suppose that $R$ is the region in the first quadrant of the plane that is bounded by the hyperbolas

$$xy = 2, \ xy = 4 \quad \text{and} \quad x^2 - y^2 = 2, \ x^2 - y^2 = 5.$$ 

Assume that $R$ is of unit density 1. Show that the polar moment of inertia of this region:

$$I_0 = \iint_R (x^2 + y^2) \, dx \, dy$$

is equal to 3.
Problem 7. (10 pts.) Let $B_a$ be the ball $x^2 + y^2 \leq a^2$. Compute

$$\iint_{B_a} \frac{1}{(2 + x^2 + y^2)^2} \, dA$$

Use the result of the previous part to show that

$$\iint_{\mathbb{R}^2} \frac{1}{(2 + x^2 + y^2)^2} \, dA = \pi/2$$
**Problem 8.** (10 pts.) Show that

\[
\int_{(1,1)}^{(2,3)} \cos(x y) \ (y \, dx + x \, dy)
\]

is well defined.

Assuming the previous part, compute

\[
\int_{(1,1)}^{(2,3)} \cos(x y) \ (y \, dx + x \, dy)
\]
Problem 9. (10 pts.)
Let $W$ be the straight solid cylinder with the base of radius $r = 3$ and positioned as on the picture on the right. (“straight” means that the line passing through $(0, 0, 0)$ and $(4, 4, 2)$ is perpendicular to the bottom and top disks that are part of the boundary of $W$.)

Let $C$ be the boundary circle of the top disk of $\partial W$ oriented as on the picture. Use Stokes Theorem to compute

$$\int_C \left( 2y^2 + e^{x^2} \right) \, dx + (3z^2) \, dy - x^2 \, dz$$
Problem 10. (10 pts.)
Let $W$ be the straight solid cylinder like in the previous problem.

Let $S$ be the lateral part of boundary of $W$ (i.e. $S$ is $\partial W$ without the bottom and top disks) Let $S$ be oriented with the normal vector pointing out of $W$.

Use Gauss-Ostrogradski (divergence) theorem to compute

$$
\iint_{S} \left[ \begin{array}{c}
\frac{x - e^{z^2} - z^2}{2z^2} \\
y + e^{z^2} \\
2z^2
\end{array} \right] \cdot \vec{n} \ dS
$$

Note that $S$ is not closed, so you need to add the two disks before using divergence theorem.
Problem 11. (10 pts.) Let $S$ be part of the sphere $x^2 + y^2 + z^2 = a^2$ that is inside the cone $z^2 = x^2 + y^2$, $z > 0$.
Find the coordinates of the centroid of $S$. 

The following boxes are strictly for grading purposes. Please do not mark.

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