Math 52 - Winter 2010 - Final Exam

Please circle your TA’s name:  
Jack Hall  Xiannan Li  David Sher

Circle the time your TTh section meets:  
10:00  11:00  1:15  2:15

Your name (print):  
Student ID:

Please sign the following:

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: _______________________________

Instructions:

• Print your name and student ID number, circle your TA’s name, the time that you attend the TTh section and sign to indicate that you accept the Honor Code.

• Read each question carefully. In order to receive full credit, please show all of your work and justify your answers.

• You have 3 hours. This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.

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Formulas you may use: Relating rectangular coordinates \(x, y, z\), cylindrical coordinates \(r, \theta, z\) and spherical coordinates \(\rho, \theta, \varphi\):

\[
\begin{align*}
z &= \rho \cos \varphi, \\
r &= \rho \sin \varphi, \\
x &= r \cos \theta = \rho \sin \varphi \cos \theta \\
y &= r \sin \theta = \rho \sin \varphi \sin \theta
\end{align*}
\]

and \( \int dx dy dz = \int r dr d\theta dz = \rho^2 \sin \varphi d\rho d\theta d\varphi \).

Trig identities: \(2 \cos^2 x = 1 + \cos 2x\), \(2 \sin^2 x = 1 - \cos 2x\).
Problem 1. (16 pts) Evaluate \( \int_{0}^{1} \int_{x}^{1} \int_{0}^{y} y \sin(xy) \, dz \, dy \, dx. \)
Problem 2. (16 pts) Consider the force field $\vec{F} = (x^2 + y^2)i + xyj$ in the plane. Find the work (circulation) done by the force $\vec{F}$ on a particle that moves along the graph of $x = y^2$ from $(0, 0)$ to $(4, 2)$. 
Problem 3. (18 pts total) Let $F(x, y, z) = (2xy + z^2, 2yz + x^2, 2zx + y^2)$.

(a) (12 pts) Find a potential for $F$.

(b) (6 pts) Let $C$ be the curve parametrized by $r(t) = (\cos t, \sin t, t)$ for $0 \leq t \leq 1$. Compute $\int_C \vec{F} \cdot \vec{T}ds$
Problem 4. (22 pts total) Let $S$ be surface parametrized by $X(s, t) = (s \sin t, s \cos t, s^2)$ for $1 \leq s \leq 5$ and $0 \leq t \leq \pi$.
(a) (16 pts) Compute the surface area of $S$.

(b) (6 pts) SET UP, but do not calculate a DOUBLE integral computing the moment of inertia about the $y$ axis of an object in the shape of the surface $S$ if the density function is $\delta(x, y, z) = \sqrt{x^2 + y^2}$. 
Problem 5.  a) (10 pts) Evaluate
\[ \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{dy \, dx}{1 + x^2 + y^2} \]

b) (10 pts) Determine all the values of the constant \( c \) such that the following integral is finite:
\[ \iint_{\mathbb{R}^2} \frac{dA}{(1 + x^2 + y^2)^c} \]
Problem 6. (18 pts) Let $C$ be the triangle formed by intersecting the plane $x + y + z = 1$ with the three coordinate planes, oriented counterclockwise as we look at it standing at the origin. Use Stokes’ Theorem to calculate

$$\int_C zdx + xdy + ydz.$$
Problem 7. (18 pts) If $\vec{F} = 2xy \vec{i} + z \cos x \vec{j} + (z^2 + xy) \vec{k}$, calculate the outwards flux of $\vec{F}$ through the part of the surface $z = x^2 + y^2$ situated below the plane $z = 4$. 
Problem 8. (16 pts) Use Green’s Theorem to evaluate \( \int_C (x^2 + y)dx + xy^2dy \) where \( C \) is the closed curve determined by \( y^2 = x \) from \((0, 0)\) to \((1, -1)\) and then \( y = -x \) from \((1, -1)\) back to \((0, 0)\).
**Problem 9.** (20 pts) Find the volume of the region inside both the sphere of radius 2 centered at the origin and the vertical cylinder of radius 1 centered at (1,0,0).
Problem 10. (18 pts total) Consider the vector field \( \vec{F}(x, y, z) = \left( x, \frac{-z}{y^2 + z^2}, \frac{y}{y^2 + z^2} \right) \).

a) (6 pts) Show that the curl of \( \vec{F} \) is zero wherever \( \vec{F} \) is defined.

b) (6 pts) Show that \( \vec{F} \) has nonzero circulation along the curve C given by \( x(t) = (\cos(t) \sin(t), \cos(t), \sin(t)) \), with for \( 0 \leq t \leq 2\pi \).

c) (6 pts) Carefully explain why the results of part a) and b) do not contradict Stokes’ theorem.
Problem 11. (18 pts) Let $\vec{F}$ be a vector field defined everywhere except at the origin. Suppose the divergence of $\vec{F}$ is 3 everywhere $\vec{F}$ is defined. Let $S_1$ and $S_2$ be the surface of the cubes centered at the origin and of sides 2 and 4 respectively. Assuming the outwards flux through $S_1$ is 5, find the outwards flux through $S_2$. Please **carefully explain** your reasoning.