Math 52
Final Exam — March 16, 2009

Name: __________________________________________

Section Leader: Josh Genauer    Lan Huang    Xiannan Li
   (Circle one)

Section Time: 10:00 11:00 1:15 2:15
   (Circle one)

• This is a closed-book, closed-notes exam. No calculators or other electronic devices will be permitted. **You have 3 hours.** If you finish early, you must hand your exam paper to a member of teaching staff.

• In order to receive full credit, please show all of your work and justify your answer. You do not need to simplify your answers unless specifically instructed to do so.

• If you need extra room, use the back sides of each page. There is also a blank page at the end of the exam for you to use. Do not unstaple or detach pages from this exam.

• Please sign the following:

> “On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.”

  **Signature: ________________________________**

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1. (8 points)

The sum of the two integrals
\[ \int_{-1}^{0} \int_{-x}^{1} \frac{y}{1+y} \, dy \, dx + \int_{0}^{1} \int_{\sqrt{x}}^{1} \frac{y}{1+y} \, dy \, dx \]
represents the integral of \( \frac{y}{1+y} \) over a region in the \( xy \)-plane.

a) Sketch the region of integration.

b) Interchange the order of integration and evaluate the integral.
2. (8 points)
   a) DIRECTLY calculate the line integral \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \) if \( C \) is the curve \( y = \sqrt{x}, \ z = x^2, \ 0 \leq x \leq 4 \), and \( \mathbf{F}(x, y, z) = (y, x, 1) \).
   
   b) Find a potential function for \( \mathbf{F} \) and use this to calculate the same integral \( \int_C \mathbf{F} \cdot \mathbf{T} \, ds \).
3. (10 points)
   a) Let $S$ be the surface in $\mathbb{R}^3$ given parametrically by $x = u \cos v, y = u \sin v, z = v$, where $0 \leq u \leq 1, 0 \leq v \leq u$. Find the area of $S$.

   b) Compute the outward flux of the vector field $F(x, y) = (x, 2y)$ across the ellipse parametrized by $x = 4 \cos t, y = 3 \sin t, 0 \leq t \leq 2\pi$. You may use any method, but explain your work.
4. (12 points)
    a) Define what it means for a region $R$ in the plane to be SIMPLY CONNECTED. For the regions below, decide whether or not they are simply connected. For each one that is NOT simply connected, show how it violates the condition.

(i) $\mathbb{R}^2$ MINUS the ray $(x, 0), x \leq 0$

(ii) $\mathbb{R}^2$ MINUS the segment $(x, 0), 0 \leq x \leq 1$

(iii) The region between the circle of radius 1 and the circle of radius 2, both centered around the origin.

(iv) $\mathbb{R}^2$ MINUS the positive quadrant, $(x, y), 0 \leq x, 0 \leq y$

b) Mark each of the statements below as True or False.

(i) Let $F$ be a vector field which is defined on all of $\mathbb{R}^2$ and suppose that its integral along a unit circle $x^2 + y^2 = 1$ is zero. Then $F$ is a conservative vector field.

(ii) If $C_1$ and $C_2$ are oriented curves, $F$ is a vector field, and the length of $C_1$ is greater than the length of $C_2$, then $\int_{C_1} F \cdot T \, ds > \int_{C_2} F \cdot T \, ds$.

(iii) If $\nabla \times F = 0$ on all of $\mathbb{R}^3$, then $F$ has a potential function on $\mathbb{R}^3$.

(iv) If $F$ is a gradient vector field, then $\int \int_S F \cdot n \, dS = 0$ for any smooth oriented surface $S$.

(v) There exists a vector field $F$ on $\mathbb{R}^3$ such that $\nabla \times F = (x, 0, 0)$.

(vi) Suppose that $S_1$ and $S_2$ are two oriented surfaces in $\mathbb{R}^3$ which have the same boundary curve $C$ and that the orientations of $S_1$ and $S_2$ induce the same orientation on $C$. Let $F$ be a smooth vector field on $\mathbb{R}^3$. Then the flux of $\nabla \times F$ through $S_1$ and through $S_2$ is the same.
5. (12 points)

a) Suppose \( W \) is a solid region bounded by a surface \( S \). Let \( n(x, y, z) = (n_1, n_2, n_3) \) be the outward unit normal to \( S \) at \((x, y, z)\), where \( n_1 = n_1(x, y, z) \), \( n_2 = n_2(x, y, z) \), and \( n_3 = n_3(x, y, z) \) are the component functions of the normal. Show that

\[
\text{Volume}(W) = \int \int_S x n_1 \; dS = \int \int_S y n_2 \; dS = \int \int_S z n_3 \; dS.
\]

b) Prove the identity

\[
\text{curl}(f \mathbf{F}) = (\nabla f \times \mathbf{F}) + f \text{curl}(\mathbf{F}),
\]

where \( f = f(x, y, z) \) is a function and \( \mathbf{F} = F(x, y, z) \) is a vector field on \( \mathbb{R}^3 \), both with continuous partial derivatives.
6. (10 points) Consider the following vector field \( F \) on \( \mathbb{R}^3 \) \( - (0, 0, 0) \)

\[
F(x, y, z) = \frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}}
\]

a) Compute \( \int \int_S F \cdot n \, dS \) (the flux of \( F \) across \( S \)), where \( S \) is the sphere \( x^2 + y^2 + z^2 = 4 \) equipped with the outward pointing unit normal.

b) Use the result in part a) to prove that there is NO vector field \( G(x, y, z) \) satisfying \( \text{curl} \, G = F \) on \( \mathbb{R}^3 \) \( - (0, 0, 0) \). Carefully state the theorem you are using and explain how it applies here.
7. (8 points) Consider the surface $S$ in $x, y, z$-space which is given in terms of spherical coordinates $\rho, \varphi, \theta$ by the equation $\rho = 1 - \cos \varphi$. (Thus $S$ is rotationally symmetric about the $z$-axis.)

a) Sketch the curve of intersection of $S$ with the $(x, z)$-plane.

b) Find the volume of the domain $D \subset \mathbb{R}^3$ which is enclosed by $S$; i.e. the volume of the region $0 < \rho \leq 1 - \cos \varphi$. 
8. (10 points) Let \( F(x, y, z) = (2y + z, 4x + y - z, 5x + 4y + z) \). Consider the oriented curve \( C \) consisting of the straight lines from \((1, 1, 0)\) to \((0, 1, 1)\), then to \((0, 0, 2)\), then to \((1, 0, 1)\), then back to \((1, 1, 0)\), traversed in that order. \( C \) bounds a parallelogram in \( \mathbb{R}^3 \). Use Stokes’ Theorem to compute the line integral of \( F \) around \( C \).
9. (10 points) Consider the circular cone with vertex at \((0, 0, 4)\), given in cylindrical coordinates by \(z = 4 - 2r\), for \(z \geq 0\). The surface does NOT include the base disc in the \(xy\)-plane. Suppose that the vector field \(F = \text{curl} \, G\) for some other vector field \(G\), where \(F = (P(x, y, z), Q(x, y, z), 14)\). Compute the flux of \(F\) through the lateral surface of the cone, with respect to the upward pointing normal. You do not need to know what \(G\), \(P\), or \(Q\) are to do this problem; find a simpler surface. You MUST clearly explain why your method is valid, including any theorems you are using.
10. (8 points) The graph above is a plot of some vector field \( F(x, y) = (P(x, y), Q(x, y)) \), oriented curves \( A, B, C \), and points \( R, T, M, N \). Use this information to answer the following questions.

(a) Is the circulation of \( F \) around the curve \( A \) positive, negative, or zero?

(b) Is the quantity \( \int_B F \cdot Tds \) positive, negative, or zero?

(c) Is the work done by \( F \) in moving a particle from \( M \) to \( N \) along \( C \) positive, negative, or zero?

(d) Is the value \( Q_x - P_y \) at \( R \) positive, negative, or zero?

(e) Is the divergence of \( F \) at \( M \) positive, negative, or zero?

(f) Is the divergence of \( F \) at \( N \) positive, negative, or zero?

(g) Would a small paddle placed at \( M \) spin clockwise or counter-clockwise?

(h) Is \( F \) a conservative vector field?
Scratch Paper