MATH 52 FINAL EXAM JUNE 5, 2009

THIS IS A CLOSED BOOK, CLOSED NOTES EXAM. NO CALCULATORS OR OTHER ELECTRONIC DEVICES ARE PERMITTED.

IF YOU NEED EXTRA SPACE, PLEASE USE THE BACK OF THE PREVIOUS PROBLEM PAGE. SO EXTRA WORK FOR PROBLEM 1 WOULD GO ON THE BACK OF THIS COVER SHEET, ETC.

MOST PROBLEMS ARE WORTH 10 POINTS. PROBLEM TEN IS 12 POINTS AND PROBLEM ELEVEN IS 8 POINTS. THE TOTAL IS 120 POINTS.

THE TERMS ‘DIFFERENTIABLE’ OR ‘TWICE DIFFERENTIABLE’ FOR FUNCTIONS AND VECTOR FIELDS MEANS THAT THEY HAVE CONTINUOUS FIRST PARTIAL DERIVATIVES OR CONTINUOUS FIRST AND SECOND PARTIAL DERIVATIVES, RESPECTIVELY, ON THEIR DOMAINS.

Please sign the following, and make sure your name is legible:

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

SIGNATURE: __________________________ NAME: __________________________

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1. The sum of the two integrals
\[
\int_{-4}^{0} \int_{-\sqrt{x}}^{2} 3x^2 \cos(y^7) \, dy \, dx + \int_{0}^{4} \int_{\sqrt{x}}^{2} 3x^2 \cos(y^7) \, dy \, dx
\]
represents the integral of \(3x^2 \cos(y^7)\) over a region \(D\) in the \(xy\)-plane.

(a) [4]. Sketch the region \(D\).

(b) [4]. Reverse the order of integration and write the integral over \(D\) as one double integral in the form
\[
\int_{c}^{d} \int_{a(y)}^{b(y)} 3x^2 \cos(y^7) \, dx \, dy.
\]
[Check that you have sketched \(D\) correctly.]

(c) [2]. Evaluate the integral in (b) or the two integrals above part (a), your choice.
2(a)[5]. Let $W$ be the solid object in the half space $x \geq 0$ bounded by paraboloids $z = 27 - 2x^2 - 2y^2$ and $z = x^2 + y^2$ and by the $yz$-plane. Assume the density of the object is given by $\delta(x, y, z) = x$. Set up but do not evaluate a triple integral in rectangular coordinates in the order $\int \int \int \, dz\,dy\,dx$ that gives the mass of the object.

(b)[5]. Set up but do not evaluate an integral in cylindrical coordinates $r, \theta, z$ that gives the mass of the same object.
3. Let \( C \) be the ellipse \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \) in the \( xy \)-plane. Let \( \vec{F} \) be the vector field given by \( \vec{F} = (3y, x) = 3y \, \vec{i} + x \, \vec{j} \).

(a) [3]. Compute the flux \( \oint_C \vec{F} \cdot \vec{n} \, ds \) where \( \vec{n} \) is the outward unit normal to curve \( C \) in the \( xy \)-plane. [Hint: Does the divergence theorem apply?]

(b) [3]. Compute the circulation \( \oint_C \vec{F} \cdot \vec{t} \, ds \), where \( \vec{t} \) is the unit tangent vector to curve \( C \) oriented clockwise. [Hint: Green’s theorem.]

(c) [4]. Using the parametrization \((4 \cos(\theta), 3 \sin(\theta))\), \( 0 \leq \theta \leq 2\pi \), for the ellipse \( C \), write down but do not evaluate an explicit integral that gives the area of one side of a fence whose base is curve \( C \) and whose height above the \( xy \)-plane is given by the formula \( 10 - y \).
4(a)[5]. Consider the change of variables \( u = xy, \ v = y/x \). Find the Jacobian determinant
\[
\frac{\partial (x,y)}{\partial (u,v)}
\]
as a function of \( u \) and \( v \). [Hint: First find the reciprocal as a function of \( x \) and \( y \).]

(b)[5]. Let \( R \) be the domain in the first quadrant of the \( xy \)-plane bounded by the hyperbolas \( xy = 1 \) and \( xy = 3 \) and by the lines \( y = 2x \) and \( y = 5x \). Use the change of variables in part (a) to compute \( \int \int_R y/x \, dx \, dy \). [Hint: \( R \) corresponds to a very simple region in the \( uv \)-plane]
5(a)[4]. The vector field \( \vec{F} = (3x^2y + z^2 + y, x^3 + 2yz + x, 2zx + y^2 + 1) \) is conservative. Find a potential function \( f(x, y, z) \) for \( \vec{F} \).

(b)[3]. If \( C \) is the curve \((t, t^2, t^3), 0 \leq t \leq 1\), compute the vector line integral \( \int_C \vec{F} \cdot d\vec{s} \), where \( \vec{F} \) is the vector field of part (a). [Use any method.]

(c)[3]. If the vector field \( \vec{F} \) in part (a) represents a force field, determine the work done by \( \vec{F} \) on a particle that moves counterclockwise once around the unit circle \( x^2 + y^2 = 1 \). Explain.
6. Let $S$ be the portion of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = -3$. [So $S$ consists of more than half of a sphere.] Let $\vec{F} = (x+3y^2+z^2, x^2+y-z^2, xy-2z)$.

(a) [2]. Calculate $\text{div}(\vec{F})$. At what points of $\mathbb{R}^3$ does your calculation hold?

(b) [8]. Compute the flux $\int \int_S \vec{F} \cdot \vec{n} \, dS$, where the unit normal is given by $\vec{n} = \frac{1}{5}(x, y, z)$ at points of $S$. Explain your work very clearly. For your own protection, NO credit will be given for attempts that try to parametrize $S$.

[Hint: $S$ is not a closed surface. But $S$ together with a disk in some plane does form a closed surface. You can also use symmetry and knowledge of certain areas to evaluate certain integrals without actually integrating.]
7. Consider the conical surface $S$ parametrized as $\vec{X}(u, v) = (usin(v), u, ucos(v))$ with $0 \leq u \leq 4$, $0 \leq v \leq 2\pi$. The vertex of $S$ is at the origin, and the axis of the cone $S$ is along the positive $y$-axis.

(a) Consider the standard normal vector $\vec{N}$ to surface $S$ produced by this parametrization. Compute $\vec{N}$ and $||\vec{N}||$, in terms of $u$ and $v$. Does $\vec{N}$ point toward or away from the axis of $S$? [$\vec{N}$ is not a unit normal. The order of the parameter variables is $(u, v)$.

(b) The surface area of $S$ is $16\sqrt{2}\pi$. Consider the centroid $(\bar{x}, \bar{y}, \bar{z})$ of $S$. [Centroid means center of mass when density $\delta = 1$. ] Two of the three centroid coordinates are obvious. Which two, and what are their values? Explain how you would calculate the third centroid coordinate in terms of an explicit integral in terms of $u$ and $v$. You do not need to evaluate.
8. In this problem, we consider the plane vector field \( \vec{F} = (P(x, y), Q(x, y)) = \left( \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right) \), which satisfies \( \partial Q/\partial x = \partial P/\partial y \) at all points where \( \vec{F} \) is defined. Parts (b) and (c) of Problem 8 are on the next page.

(a)[4]. If \( C \) is the unit circle \( x^2 + y^2 = 1 \) oriented counterclockwise, calculate \( \oint_C \vec{F} \cdot d\vec{s} \). Why doesn’t Green’s theorem give the result \( \oint_C \vec{F} \cdot d\vec{s} = 0 \)?
8(b)[3]. Sketch the closed oriented polygonal curve $C'$ with vertices 

$$(3, 0), (3, -3), (0, -3), (-3, 0), (0, 3), (3, 0)$$

in that order. Explain how Green’s theorem and the result of part 8(a) on the previous page determines $\int_C \vec{F} \cdot d\vec{s}$, and give the value.

8(c)[3]. If $C''$ is the square in the first quadrant $x > 0, y > 0$ with vertices $(1, 1), (2, 1), (2, 2), (1, 2)$ oriented counterclockwise, evaluate $\int_{C''} \vec{F} \cdot d\vec{s}$ and explain your answer. [Hint: Use any method. Reread the first paragraph on the previous page.]
9(a)[3]. Consider the plane $P$ with equation $2x + y - 2z = 4$. Find a unit normal vector $\vec{n}$ to plane $P$ that points ‘up’, that is, with positive $z$-coordinate.

(b)[2]. Consider a simple closed curve $C$ on plane $P$ that bounds a region $D$ on plane $P$. Choose the unit normal $\vec{n}$ to $D$ as in part (a). If $C$ is oriented with unit tangent vectors $\vec{t}$ so as to be consistent with the conditions of Stokes theorem for $C$ and $D$, should vector $\vec{t} \times \vec{n}$ point into or out of the region $D$ on plane $P$? Explain. [Be careful.]

(c)[5]. Let $\vec{F} = (az, bx, cy)$, where $a, b, c$ are constants. Calculate $\text{curl}(\vec{F})$. With orientations $\vec{n}$ and $\vec{t}$ chosen as in parts (a) and (b), so that Stokes theorem applies, find necessary and sufficient conditions on constants $a, b, c$ that guarantee $\oint_C \vec{F} \cdot \vec{t} \, ds > 0$. 

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10(a)[6]. Which of the regions described below are simply connected? Answer YES or NO. Score is Right minus 1/2 Wrong, with a minimum of 0. There are six regions.

(i). $R^3$ with the $y$ axis removed.

(ii). $R^3$ with the circle $x^2 + y^2 = 1$, $z = 0$ in the $xy$-plane removed.

(iii). $R^3$ with the origin removed.

(iv). $R^2$ with the origin removed.

(v). The half plane $x \geq 0$ in $R^2$.

(vi). The inside of a square in $R^2$ described as all $(x, y)$ with $0 < x < 1$, $0 < y < 1$.

(b)[6]. Write T (true) or F (false) in the margin to the left of each statement below. Score is Right minus 1/2 Wrong, with a minimum of 0. There are six statements.

(i). If $\text{curl}(\vec{F}) = \vec{0}$ on all of $R^3$, then $\oint_C \vec{F} \cdot d\vec{s} = 0$ for all closed curves $C$ in $R^3$.

(ii). There exists a vector field $\vec{G}$ on $R^3$ with $\text{curl}(\vec{G}) = (y, x, z)$.

(iii). If $\vec{F}$ is a differentiable vector field on all of $R^3$ and if $\text{div}(\vec{F}) = 1$ at all points of a solid region $W$ in $R^3$ and if $S$ is the boundary of $W$ then the outward flux $\int_S \vec{F} \cdot \vec{n} \, dS$ always equals the surface area of $S$.

(iv). If $\vec{F}$ is a differentiable vector field on all of $R^3$ and if $\oint_C \vec{F} \cdot d\vec{s} = 0$ for all simple closed curves $C$ in $R^3$, then $\vec{F} = \vec{0}$.

(v). If $\vec{F}$ is a twice differentiable vector field on $R^3 - (0, 0, 0)$, that is, everywhere except the origin, and if $S$ is the sphere of radius 1 and center $(0, 0, 0)$, then $\int_S \text{curl}(\vec{F}) \cdot d\vec{S} = 0$.

(vi). If $\vec{F}$ is a conservative vector field, differentiable on all of $R^3$, and if $S$ is a closed oriented surface, that is, with no boundary, then $\int_S \vec{F} \cdot d\vec{S} = 0$. 

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11(a)[4]. Let $f(x, y, z)$ be a function on $\mathbb{R}^3$ and $\vec{F}(x, y, z)$ a vector field on $\mathbb{R}^3$, both with continuous partial derivatives everywhere. There is a ‘product rule’ for $\text{div}(f\vec{F})$, expressing this divergence as a sum of two terms. Prove this product rule.

[You don’t need to be told the rule. Just work out $\text{div}(f\vec{F})$ yourself and see the rule. As a hint, one of the two terms is a dot product.]

(b)[4]. If $\vec{F} = (P, Q, R)$ is a twice differentiable vector field in a region $W$ in $\mathbb{R}^3$, show that $\text{div}(\text{curl}(\vec{F})) = 0$ in $W$. Be clear about how the hypotheses justify your work.
The graph on the next two pages ‘plots’ a vector field $\vec{F}(x, y) = (P(x, y), Q(x, y))$, oriented curves $A, B, C$, and points $M, N, R, T$. Answer the following ten questions. There are two copies of the graph so you can tear off the last page while contemplating the questions. Score is, as usual, Right $-\frac{1}{2}$ Wrong, with a minimum of 0. [One assumes $\vec{F}$ is differentiable.]

(i). Is the circulation of $\vec{F}$ around curve $B$ positive, negative, or 0?

(ii). If a unit normal $\vec{n}$ is chosen to curve $C$ pointing down through the letter $C$ at points near that letter, is the quantity $\int_C \vec{F} \cdot \vec{n} \, ds$ positive, negative, or 0?

(iii). Is the quantity $\int_A \vec{F} \cdot d\vec{s}$ positive, negative, or 0?

(iv). Is the value of $\frac{dQ}{dx} - \frac{dP}{dy}$ at $T$ positive, negative, or 0?

(v). Is the quantity $\frac{dP}{dx} + \frac{dQ}{dy}$ at $R$ positive, negative, or 0?

(vi). Is the divergence of $\vec{F}$ at $M$ positive, negative, or 0?

(vii). If $D$ is the plane region bounded by curve $A$, is the quantity $\iint_D \text{div}(\vec{F}) \, dx \, dy$ positive, negative, or 0?

(viii). If $\vec{F}$ is a fluid flow velocity field, would a small paddle at $N$ spin clockwise, counterclockwise, or not at all?

(ix). If $\vec{F}$ is a force field, is the total work done by $\vec{F}$ on a particle moving along curve $C$ positive, negative, or 0?

(x). Is it possible that $\vec{F} = \nabla f$ for some function $f(x, y)$?