1. (20 pts) Consider the autonomous equation

\[ y' = (y^2 - 9)(y^2 - 1)(y^2 + 1). \]

Determine the stationary points and classify each as stable or unstable. Sketch in the \((t, y)\)-plane the solution curves passing through the points \((0, k)\), where \(k = 0, \pm 1, \pm 2, \pm 3, \pm 4\).

**Solution:** The critical points of this equation are at \(y = \pm 1\) and \(\pm 3\). The direction field in the \((t, y)\)-plane has positive slopes when \(y > 3\), and also when \(-1 < y < +1\) and \(y < -3\); the slopes are negative when \(1 < y < 3\) and \(-3 < y < -1\). From this information you can see that \(y = -3\) and \(+1\) are stable and \(y = -1\) and \(+3\) are unstable. The solution curves below \(y = -3\) are asymptotic to this line as \(t \to +\infty\), but tend to \(-\infty\) as \(t\) decreases. The solutions curves in the strip \(-3 < y < -1\) converge to the upper asymptote as \(t \to -\infty\) and to the lower asymptote as \(t \to +\infty\); the same is true for the solution curves in the strip \(1 < y < 3\). On the other hand, the solution curves in \(-1 < y < +1\) converge to the lower asymptote as \(t \to -\infty\) and to the upper asymptote as \(t \to +\infty\). Finally, the solution curves above \(y = +3\) converge to this line asymptotically as \(t \to -\infty\) but tend to \(+\infty\) as \(t\) increases.

2. (20 pts) Suppose a large vat initially contains 100 liters of pure water. Starting at time \(t = 0\), a brine mixture with a concentration of 10 grams of salt per liter is poured in at the rate of 1 liter per minute, and simultaneously, the liquid solution is allowed to flow out from a tap at exactly the same rate of 1 liter per minute. Assume that the mixture in the vat is constantly being stirred so that the brine maintains a uniform concentration. At what time will there be exactly 500 grams of salt in the vat?
Solution: The differential equation for the function $S(t)$ is

$$S'(t) = 10 - \frac{S(t)}{100}.$$  

(The units are grams/minute: the 10 is 10 grams/liter times 1 liter/minute, while the $S(t)/100$ is $S(t)$ grams/100 liters times 1 liter/minute.) This is separable: we write

$$\frac{S'(t)}{10 - S(t)/100} = 1 \Rightarrow -100 \ln(10 - S(t)/100) = t + C$$

and hence

$$S(t) = 1000 - C' e^{-t/100}.$$  

To match the initial condition $S(0) = 0$ we need $C' = 1000$, so that

$$S(t) = 1000(1 - e^{-t/100}).$$  

Finally, $S(t_0) = 500$ implies $e^{-t_0/100} = 1/2$, i.e.

$$t_0 = 100 \ln 2$$

is the time at which there are 500 grams of salt in the brine mixture.

3. (a) (10 pts) Determine whether the differential

$$(\cos x + \cos x \cos^2 y) \, dx - 2 \sin x \sin y \cos y \, dy = 0$$

is exact. If it is, find the complete set of solutions.

Solution: This is exact since

$$\frac{\partial}{\partial y}(\cos x + \cos x \cos^2 y) = -2 \cos x \sin y \cos y = \frac{\partial}{\partial x}(-2 \sin x \sin y \cos y).$$

Taking antiderivatives, we see that the solution curves of this equation are given by the level curves of the function

$$F(x, y) = \sin x(1 + \cos^2 y).$$
(b) (10 pts) Find and sketch the solution curve of this equation which passes through the point \((0, 1)\).

**Solution:** Since \(F(0, 1) = \sin 0(1 + \cos^2 1) = 0\), the correct solution curve is the one given by \(F(x, y) = 0\). But \(F = 0\) implies \(\sin x = 0\) since \(1 + \cos^2 y\) never vanishes. There are infinitely many vertical lines on which \(\sin x\) vanishes, but the precise solution curve passing through this point is the vertical line \(x = 0\), i.e. the \(y\)-axis.

4. (a) (5 pts) Suppose that the constant coefficient linear equation

\[ y'' + py' + qy = 0 \]

has a solution \(y(t)\) which satisfies \(y(t + 1) = y(t)\) for every \(t\). Use this information to find the values of the constants \(p\) and \(q\).

**Solution:** The solutions of any constant coefficient equation of second order are (with the usual exceptions for repeated roots) linear combinations of the (possibly complex) exponential functions \(e^{\lambda t}\), where

\[ \lambda^2 + p\lambda + q = 0. \]

The roots are

\[ \lambda_1, \lambda_2 = -\frac{p}{2} \pm \frac{1}{2} \sqrt{p^2 - 4q}. \]

Exponential functions where the exponent is real are *never* periodic (check this!), and furthermore, complex exponentials are only periodic when the damping coefficient \(p\) is zero (otherwise we would get a damped oscillation with decreasing or increasing amplitudes). Using that \(p = 0\), we see that

\[ \lambda_1, \lambda_1 = \pm \sqrt{q} i \]

and the corresponding (real-valued) solutions are

\[ \sin(\sqrt{q} t), \quad \cos(\sqrt{q} t). \]
Letting $y(t)$ be a linear combination of these, then $y(t+1) = y(t)$ is true for all $t$ if and only if $\sqrt{q}$ is an integer multiple of $2\pi$. Thus, necessarily,

$$\sqrt{q} = 2\pi k \Rightarrow q = 4\pi^2 k^2 \quad k = 0, 1, 2, \ldots.$$  

(Note: $k = 0$, or equivalently $q = 0$, corresponds to the equation $y'' = 0$, which has the solution $y(t) \equiv 1$, which certainly satisfies $y(t+1) = y(t)$ for all $t$.)

(b) (10 pts) Find the solutions $y_1(t)$, $y_2(t)$ for the ODE $y'' - y = 0$ such that

$$y_1(0) = 1, \quad y'_1(0) = 0, \quad y_2(0) = 0, \quad y'_2(0) = 1.$$  

In addition, calculate the Wronskian $W(t)$ associated to these two functions.

**Solution:** The polynomial $\lambda^2 - 1$ has roots $\pm 1$, and hence one basis of solutions is

$$u_1(t) = e^t, \quad u_2(t) = e^{-t}.$$  

However, these do not satisfy the initial conditions. The correct basis of solutions is

$$y_1(t) = \frac{1}{2}(u_1(t) + u_2(t)) = \cosh t = \frac{e^t + e^{-t}}{2},$$  

$$y_2(t) = \frac{1}{2}(u_1(t) - u_2(t)) = \sinh t = \frac{e^t - e^{-t}}{2}.$$  

4 (cont.) (c) (10 pts) Find the general solution $y(t)$ of the ODE

$$y'' - 3y' + 2y = 2t + \cos t.$$  

Then, find the unique solution which satisfies $y(0) = 0$, $y'(0) = 0$.

**Solution:** The polynomial equation $\lambda^2 - 3\lambda + 2 = 0$ has solutions $\lambda = 1, 2$, so a basis for the set of homogeneous solutions is

$$y_1(t) = e^t, \quad y_2(t) = e^{2t}.\]
We use the method of undetermined coefficients to find a particular solution of the inhomogeneous equation.

Guessing $At + B + C \sin t + D \cos t$ leads to the simultaneous linear equations

$$2A = 2, \quad -3A + 2B = 0,$$

and

$$C + 3D = 0, \quad -3C + D = 1.$$ 

These have the solutions

$$A = 1, \quad B = 3/2, \quad \text{and} \quad C = -3/10, \quad D = 1/10.$$ 

Hence the general solution to this inhomogeneous ODE is

$$y(t) = t + \frac{3}{2} - \frac{3}{10} \sin t + \frac{1}{10} \cos t + c_1 e^t + c_2 e^{2t}.$$ 

We now compute

$$y(0) = 3/2 + 1/10 + c_1 + c_2 = 0, \quad y'(0) = 1 - 3/10 + c_1 + 2c_2 = 0,$$

which leads to the solutions $c_1 = -5/2, c_2 = 9/10$. The unique solution to this IVP is

$$y(t) = t + \frac{3}{2} - \frac{3}{10} \sin t + \frac{1}{10} \cos t - \frac{5}{2} e^t + \frac{9}{10} e^{2t}.$$

5. (15 pts) Suppose that $y_1(t)$ and $y_2(t)$ are two solutions of the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

where $p(t)$ and $q(t)$ are differentiable functions defined on the entire real line.

Suppose that $y_1(1) = y_2(1) = 0$. Give a careful explanation of why it is then necessarily true that there exist constants $c_1$ and $c_2$ so that $c_1y_1(t) + c_2y_2(t) = 0$ for all $t$.

**Solution:** As proved in class, the Wronskian determined by these two solutions,

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$
is either identically zero or never vanishes for any value of \( t \). However, our hypothesis assures us that \( W(1) = 0 \), so that \( W(t) = 0 \) for all \( t \). In other words, the solutions \( y_1 \) and \( y_2 \) are dependent. To really spell things out, let us describe why this implies the existence of such constants.

The general structure theory for homogeneous linear solutions of second order, with differentiable coefficient functions defined on the whole line assures us that there exists a basis of solutions of this equation, \( z_1(t) \) and \( z_2(t) \). For example, one way to choose this basis is to solve the initial value problem with initial conditions

\[
    z_1(1) = 1, \quad z'_1(1) = 0, \quad z_2(1) = 0, \quad z'_2(1) = 1.
\]

The solutions \( y_1 \) and \( y_2 \) in the statement of the problem must be linear combinations of these, i.e.

\[
    y_1 = a_1 z_1 + a_2 z_2, \quad y_2 = b_1 z_1 + b_2 z_2.
\]

Evaluating at \( t = 1 \) shows that \( a_1 = b_1 = 0 \), or in other words, both \( y_1 \) and \( y_2 \) are constant multiples of the same function \( z_2 \). This clearly implies that \( c_1 y_1 + c_2 y_2 = 0 \) for \( c_1 = b_2, \ c_2 = -b_1 \).