1. (20 points) (a) Find the maximal interval of definition of the solution to the initial value problem

\[ \ln \left( \frac{t}{2} - 1 \right) y' = y + e^{2t}, \quad y(3) = 1. \]

This is a linear equation, so we use Theorem 2.3.1 from the book. Put into standard form:

\[ y' - \left( \frac{1}{\ln(t/2 - 1)} \right) y = \frac{e^{2t}}{\ln(t/2 - 1)} g(t). \]

The maximal interval of definition is the maximal open interval \( I \) containing \( t = 3 \) on which both \( p(t) \) and \( g(t) \) are continuous. For \( p(t) \), we require that \( t/2 - 1 > 0 \), i.e., \( t > 2 \), so that the logarithm is defined; furthermore, we need that \( t/2 - 1 \neq 1 \), i.e., \( t \neq 4 \), since otherwise we have division by zero. The same clearly holds for \( g(t) \). Therefore, \( p(t) \) and \( g(t) \) are simultaneously continuous on \( 2 < t < 4 \) and \( t > 4 \). The initial condition at \( t = 3 \) hence gives \( I = (2, 4) \).

(b) Find all values \((t_0, y_0)\) such that a unique solution is guaranteed for the initial value problem

\[ y' = |y - t|, \quad y(t_0) = y_0. \]

(Hint: break into different cases.)

**Solution 1:** This is a nonlinear equation, so we use Theorem 2.3.2 from the book. Then

\[ f(t, y) = |y - t|, \quad \partial f = \left\{ \begin{array}{ll} +1 & \text{if } y > t, \\ -1 & \text{if } y < t. \end{array} \right. \]

Clearly, \( f \) is continuous everywhere in the \( ty \)-plane, while \( f_y \) is continuous in each open half-space \( y > t \) and \( y < t \); however, it is discontinuous (in fact, undefined) at \( y = t \). Therefore, the theorem applies only within each half-space and uniqueness (in some local neighborhood) is guaranteed for all \((t_0, y_0)\) such that \( y_0 \neq t_0 \).

**Solution 2:** Alternatively, we can write this as

\[ y' = \begin{cases} y - t & \text{in } R_1 = \{(t, y) \mid y \geq t\}, \\ t - y & \text{in } R_2 = \{(t, y) \mid y \leq t\} \end{cases} \]

and apply the linear theorem to each case. Fix \( y_0 \). Then if \( t_0 < y_0 \), applying the theorem in \( R_1 \) gives uniqueness for all \( t < y \); similarly, if \( t_0 > y_0 \), then applying the theorem in \( R_2 \) gives uniqueness for all \( t > y \). Note that we have excluded \( t_0 = y_0 \) since there is no open interval containing it in either case. Therefore, uniqueness is guaranteed for all \( t_0 \neq y_0 \).

**Advanced:** Actually, the problem is quite subtle and there is a unique solution for all \((t_0, y_0)\). Solve each equation in \( R_{1,2} \) above and impose the initial condition \( y_0 = t_0 \) to get

\[ y_1 = t + 1 - e^{t-t_0} \quad \text{in } R_1, \]

\[ y_2 = t - 1 + e^{-(t-t_0)} \quad \text{in } R_2, \]

where \( y_i \) is the solution in \( R_i \). Check that \( y_1 \) is valid (i.e., satisfies \( y_1 \geq t \) on \( t \leq t_0 \); similarly, \( y_2 \) is valid on \( t \geq t_0 \). So \( y_{1,2} \) overlap only at \( t = t_0 \), where, in fact, they both agree: \( y_{1,2}(t_0) = y_0 \) and \( y'_{1,2}(t_0) = |y_0 - t_0| = 0 \). Thus, the solution is unique and can be “glued together” as

\[ y = \begin{cases} y_1 & \text{if } t \leq t_0, \\ y_2 & \text{if } t > t_0. \end{cases} \]
2. (20 points) Consider the system of differential equations

\[ \mathbf{x}' = A \mathbf{x}, \quad A = \begin{bmatrix} 2 & \alpha \\ 1 & 0 \end{bmatrix}. \]

(a) Find the eigenvalues of \( A \). For what values of \( \alpha \) is the origin a saddle point?

The characteristic polynomial is \( \det(A - \lambda I) = \lambda^2 - \text{Tr}(A) \lambda + \det(A) = (\lambda - \lambda_1)(\lambda - \lambda_2) \). For a homogeneous system to have a saddle point, it is necessary and sufficient to have real eigenvalues with opposite signs. Hence, the system has a saddle point if and only if it has real roots and \( \lambda_1 \lambda_2 = \det(A) < 0 \). Thus the discriminant has to be positive, \( \text{Tr}(A)^2 - 4 \det(A) = 4 + 4\alpha > 0 \), and the determinant has to be negative, \( \det(A) = -\alpha < 0 \). Hence, to have a saddle point, we need to have \( \alpha > 0 \).

(b) Let \( \alpha = 3 \). Find the general solution.

For \( \alpha = 3 \), the roots of the characteristic polynomial are \( \lambda_1 = 3 \) and \( \lambda_2 = -1 \). Suppose \( v_1 = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \) is an eigenvector for \( \lambda_1 \). Hence, we have

\[ \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

which implies \( -s_1 + 3s_2 = 0 \), so we can choose \( v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \) to be an eigenvector for \( \lambda_1 \). Now assume \( v_2 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \) is an eigenvector for \( \lambda_1 = -1 \). To find \( v_2 \), we need to solve the following

\[ \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

which implies \( r_1 + r_2 = 0 \), so we can choose \( v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) to be an eigenvector for \( \lambda_2 \). The general solution in the case of real different eigenvalues, is given by \( \mathbf{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 \):

\[ \mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]
(c) Let $\alpha = 3$. For ease of reference, the corresponding system is

$$x' = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} x.$$ 

Draw a phase portrait that clearly shows the asymptotic behavior as $t \to \infty$. (Hint: it may help to draw a few slope vectors.)
3. (20 points) Consider the system of differential equations

\[ x'(t) = -y \]
\[ y'(t) = 4x. \]

(a) Find the general real-valued solution.

We begin by writing the equation in matrix form: Let \( \vec{z} := \begin{pmatrix} x \\ y \end{pmatrix} \), and we get

\[ \vec{z}' = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix} \vec{z}. \]

Then we compute the characteristic polynomial in order to find the eigenvalues:

\[
\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 4 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0.
\]

Thus the eigenvalues are \( \pm 2i \). Picking one at random, say \( \lambda = 2i \), we find an eigenvector, by solving

\[
(A - \lambda I)v = 0,
\]
\[
\begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \vec{v} = 0.
\]

We may choose our eigenvector to be \( \begin{pmatrix} 1 \\ -2i \end{pmatrix} \). The other eigenvector can then be chosen to be the complex conjugate \( \begin{pmatrix} 1 \\ 2i \end{pmatrix} \).

The general solution, in complex form, is thus given by

\[ \vec{z}(t) = c_1 e^{2it} \begin{pmatrix} 1 \\ -2i \end{pmatrix} + c_2 e^{-2it} \begin{pmatrix} 1 \\ 2i \end{pmatrix}. \]

To find the real-valued solutions, we expand the first term using Euler’s formula:

\[
e^{2it} \begin{pmatrix} 1 \\ -2i \end{pmatrix} = \begin{pmatrix} \cos(2t) + i\sin(2t) \\ -2i \cos(2t) + 2 \sin(2t) \end{pmatrix} = \begin{pmatrix} \cos(2t) \\ 2 \sin(2t) \end{pmatrix} + i \begin{pmatrix} \sin(2t) \\ -2 \cos(2t) \end{pmatrix}.
\]

Thus, we can write down the general real solution:

\[ \vec{z}(t) = c_1 \begin{pmatrix} \cos(2t) \\ 2 \sin(2t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(2t) \\ -2 \cos(2t) \end{pmatrix}. \]
(b) Find the particular solution passing through \( x(0) = 1, y(0) = 0 \). Draw a component plot.

Plugging in \( t = 0 \) to the general form, we have that

\[
\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -2 \end{pmatrix}.
\]

Equating components gives \( 1 = x(0) = c_1 \) and \( 0 = y(0) = -2c_2 \), so \( c_1 = 1, c_2 = 0 \), and the specific solution is given by

\[
x(t) = \cos(2t), y(t) = 2\sin(2t).
\]

(c) Identify the type of critical point at the origin. If it is a center or a spiral point, state the direction of rotation (clockwise or counterclockwise) in the \( xy \)-plane.

The node at the origin is a [center], since the eigenvalues are complex and have 0 real part.

Evaluating the direction field at an arbitrary point, say \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \), we find that

\[
A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix},
\]

so the center turns in the [counterclockwise] direction.
4. (24 points) Consider the system of differential equations

\[ \mathbf{x}' = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \mathbf{x}. \]

(a) Find the general solution.

Let

\[ A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}. \]

Then

\[ \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ -4 & 4 - \lambda \end{bmatrix} = (\lambda - 2)^2, \]

and \( A \) has only one eigenvalue \( \lambda = 2 \). Let

\[ v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \]

Solving

\[ \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \]

we have

\[ v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \]

Since \( \lambda = 2 \),

\[ A - \lambda I = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}. \]

Let

\[ w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \]

Solving

\[ \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \]

we have

\[ w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

Therefore, we have

\[ x_1(t) = e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \]

and

\[ x_2(t) = te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]

The general solution of the system is

\[ x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \left( te^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right). \]
(b) Draw a phase portrait that clearly shows the asymptotic behavior as $t \to \infty$. (Hint: it may help to draw a few slope vectors.)

(c) Identify the type of critical point at the origin and its stability.

The origin $(0,0)$ is an unstable improper node.
5. (16 points)  
(a) Rewrite the second-order linear differential equation

\[ x'' + 2x' - e^t x = 1 \]

as a first-order system of two equations.

Let’s introduce two new variables:

\[ x_1 = x, \quad x_2 = x'. \]

Then the equation \( x'' + 2x' - e^t x = 1 \) can be written as

\[
\begin{cases}
  x_1' = x_2 \\
  x_2' + 2x_2 - e^t x_1 = 1
\end{cases}
\]

i.e.,

\[
\begin{cases}
  x_1' = x_2 \\
  x_2' = e^t x_1 - 2x_2 + 1
\end{cases}
\]

So the system of equations is

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ e^t & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

(b) Without computing eigenvalues/eigenvectors, show that

\[
\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

form a fundamental pair of solutions for

\[
\begin{pmatrix} x' \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

First we need check \( \vec{x}_1 \) and \( \vec{x}_2 \) are solutions to the equation.

\[
\begin{align*}
\vec{x}_1' &= 2e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \\
A\vec{x}_1 &= e^{2t} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 4 \\ 2 \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
\vec{x}_2' &= 3e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{3t} \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \\
A\vec{x}_1 &= e^{3t} \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = e^{3t} \begin{pmatrix} 3 \\ 3 \end{pmatrix}.
\end{align*}
\]

We find that

\[
\begin{align*}
\vec{x}_1' &= A\vec{x}_1, \\
\vec{x}_2' &= A\vec{x}_2.
\end{align*}
\]

So \( \vec{x}_1 \) and \( \vec{x}_2 \) are solutions to the equation \( \vec{x}' = A\vec{x} \). (Notice that the problem requires not finding eigenvalues and eigenvectors, we should check them directly.) To see that \( \vec{x}_1 \) and \( \vec{x}_2 \) form a fundamental set of solutions, we need compute the Wronskian:

\[
W[\vec{x}_1, \vec{x}_2](t) = \det \begin{pmatrix} 2e^{2t} & e^{3t} \\ e^{2t} & e^{3t} \end{pmatrix} = e^{5t} \neq 0.
\]

Therefore, \( \vec{x}_1 \) and \( \vec{x}_2 \) form a fundamental set of solutions.