1. Solve the system of equations $\frac{dx}{dt} = 3x - y$, $\frac{dy}{dt} = 3y - x$ with the initial condition $x(0) = 2$ and $y(0) = 0$. [Your final answer must be expressed in terms of real functions.]

**Solution.** Letting $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, we must solve

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \vec{x} = A\vec{x}.$$ 

To find the eigenvalues, we solve $\det(A - \lambda I) = (3 - \lambda)^2 - 1 = 0$, so we have $\lambda_1 = 2$ and $\lambda_2 = 4$.

The eigenspace for $\lambda_1 = 2$ is the nullspace of $A - 2I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, spanned by the eigenvector $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The eigenspace for $\lambda_2 = 4$ is the nullspace of $A - 4I = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$, spanned by the eigenvector $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Thus the general solution is

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$ 

and the initial conditions $x(0) = 2$, $y(0) = 0$ give $c_1 + c_2 = 2$, $-c_1 + c_2 = 0$. Thus $c_2 = c_2 = 1$ and the solution is

$$\vec{x}(t) = e^{2t} + e^{4t}, \quad y(t) = e^{2t} - e^{4t}. \quad \Box$$

2. (a) Find the general solution of $\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \vec{x}$. [Your final answer must be expressed in terms of real functions.]

**Solution.** Letting $A = \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix}$, we first obtain the eigenvalues by solving $\det(A - \lambda I) = (1 - \lambda)^2 + 9 = 0$, obtaining $\lambda = 1 \pm 3i$.

The eigenspace for $\lambda = 1 + 3i$ is the nullspace of $A - (1 + 3i)I = \begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix}$, which is spanned by the eigenvector $\vec{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$. A complex solution is given by $\vec{x} = e^{\lambda t} \vec{v}$, and we now use the fact that its real and imaginary parts give two linearly independent real solutions.

$$e^{\lambda t} \vec{v} = e^{(1+3i)t} \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^t \left( \cos 3t + i \sin 3t \right) \begin{pmatrix} 1 \\ -i \end{pmatrix} = e^t \begin{pmatrix} \cos 3t + i \sin 3t \\ \sin 3t - i \cos 3t \end{pmatrix}$$

$$= e^t \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + i e^t \begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix}$$

Hence the general solution of the system is

$$\vec{x}(t) = c_1 e^t \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix}. \quad \Box$$
(b) Sketch a phase portrait of this system, and state the stability of the critical point at the origin.

*Solution.* The eigenvalues are $1 \pm 3i$, complex with positive real part, hence the phase portrait has a single critical point at the origin which is unstable, and the phase portrait is an outwards spiral. To determine its direction, we can for example substitute $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and find the direction of the trajectory through this point. We obtain there $\frac{dx}{dt} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, i.e. the trajectory points up and right across the positive $x$-axis. Hence it is an anticlockwise outwards spiral.

3. Consider the equation $y + (4x + 5y) \frac{dy}{dx} = 0$.

(a) Find $\alpha$ so that after multiplying this equation by $y^\alpha$ it becomes exact.

*Solution.* Multiplying by $y^\alpha$ gives

$$y^{\alpha+1} + y^\alpha (4x + 5y) \frac{dy}{dx} = 0,$$

which is of the form $M(x,y) + N(x,y) \frac{dy}{dx} = 0$, where $M(x,y) = y^{\alpha+1}$ and $N(x,y) = y^\alpha(4x + 5y)$. This will be exact when $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, i.e.

$$(\alpha+1)y^\alpha = 4y^\alpha.$$

This equality holds for all $y$ if and only if $\alpha = 3$; so the equation is exact precisely for $\alpha = 3$.

(b) Find the general solution of this equation. [You can leave your answer as an implicit relation between $x$ and $y$.]

*Solution.* Setting $\alpha = 3$, our equation becomes

$$y^4 + y^3 (4x + 5y) \frac{dy}{dx} = 0.$$

We will find a function $F(x,y)$ such that $\frac{\partial F}{\partial x} = M = y^4$ and $\frac{\partial F}{\partial y} = N = y^3(4x + 5y)$. From $\frac{\partial F}{\partial x} = y^4$ we obtain

$$F(x,y) = xy^4 + h(y)$$

for some function $h$ of $y$. Substituting this expression for $F$ into $\frac{\partial F}{\partial y} = y^3(4x + 5y)$ gives

$$4xy^3 + h'(y) = 4xy^3 + 5y^4,$$

and hence $h'(y) = 5y^4$. We may take $h(y) = y^5$ and then we have $F(x,y) = xy^4 + y^5$ as our desired function. The solution curves of our differential equation are the level curves of $F(x,y) = xy^4 + y^5$, i.e. $xy^4 + y^5 = c$ for constants $c$.

4. Consider the autonomous equation $\frac{dy}{dt} = e^{-y}(1 - y^2)$. [Warning: This equation can not be solved explicitly.]

(a) Find the equilibria of this system and classify them as stable or unstable.

*Solution.* The equilibria occur when $\frac{dy}{dt} = 0$ precisely when $e^{-y}(1 - y^2) = 0$. But $e^{-y}$ is never zero, hence we have only $1 - y^2 = 0$, i.e. $y = \pm 1$.

To determine the stability of these equilibria, we can sketch a direction graph, or recall that in general, at an equilibrium point of $\frac{dy}{dt} = f(y)$, i.e. where $f(y) = 0$, the equilibrium is stable if $f'(y) < 0$ and unstable if $f'(y) > 0$. Here we have

$$f'(y) = \frac{d}{dy}(e^{-y})(1 - y^2) + e^{-y} \frac{d}{dy}(1 - y^2) = -e^{-y}(1 - y^2) + e^{-y}(-2y) = -e^{-y}(y^2 - 2y - 1).$$

We obtain $f'(-1) = 2e^1 > 0$ and $f'(1) = -2e^{-1} < 0$, so $y = -1$ is unstable and $y = 1$ is stable.

(b) Let $y$ be a solution of this system with $y(0) = -53$. Determine the concavity of $y$ for $t \geq 0$. That is, for what values of $t \geq 0$ is $y$ concave up, and for what values of $t \geq 0$ is $y$ concave down. [Hint: Compute $\frac{d^2y}{dt^2}$, and express your answer as a function of $y$ alone.]
Solution. First, we note that the trajectory beginning at \( y(0) = -53 \) has \( y(t) < -53 \) for all \( t \). This is clear from sketching a direction field, or else from noting that \( \frac{dy}{dt} = e^{-y}(1 - y^2) \) is negative for all \( y < -1 \). So the solution \( y(t) \) is a decreasing function, and for all \( t \geq 0 \), we have \( y(t) \leq -53 \).

The solution curve will be concave up when \( \frac{d^2 y}{dt^2} > 0 \), and concave down when \( \frac{d^2 y}{dt^2} < 0 \), so we compute this second derivative. In general, differentiating the equation \( \frac{dy}{dt} = f(y) \) with respect to \( t \) gives \( \frac{d^2 y}{dt^2} = \frac{d}{dt} f(y) = \frac{dy}{dt} f'(y) \). We have \( f(y) = e^{-y}(1 - y^2) \) and we computed \( f'(y) = e^{-y}(y^2 - 2y - 1) \) above, so

\[
\frac{d^2 y}{dt^2} = f'(y) = e^{-2y} (1 - y^2) (y^2 - 2y - 1)
\]

This is a smooth function of \( y \), and its zeroes occur when \( e^{-2y} = 0 \) (never), or \( 1 - y^2 = 0 \) i.e. \( y = \pm 1 \), or when \( y^2 - 2y - 1 = 0 \), i.e. \( y = 1 \pm \sqrt{2} \). Thus the sign of \( \frac{d^2 y}{dt^2} \) is the same for all \( y < -1 \), and substituting any value of \( y < -1 \) shows that it is negative. Since our curve \( y(t) \) always lies below \( y = -1 \), we always have \( \frac{d^2 y}{dt^2} < 0 \), and hence the solution curve is concave down for all \( t \geq 0 \).

5. Let \( A \) be a \( 2 \times 2 \) matrix, \( \vec{v}_1, \vec{v}_2 \) be two linearly independent vectors, and \( \lambda \in \mathbb{R} \) be such that \( A\vec{v}_1 = \lambda \vec{v}_1 \) and \( A\vec{v}_2 = \lambda \vec{v}_2 + \vec{v}_1 \).

(a) Write down two linearly independent solutions of the ODE \( \frac{d^2 x}{dt^2} = A\vec{x} \). Directly verify that your solutions are indeed linearly independent. [You don’t have to verify that they’re solutions, since we did this in class.]

Solution. Two linearly independent solutions are

\[
\vec{x}_1(t) = e^{\lambda t} \vec{v}_1, \quad \vec{x}_2(t) = e^{\lambda t} (t\vec{v}_1 + \vec{v}_2).
\]

To see they are linearly independent, suppose some linear combination of them is zero, i.e.

\[
c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} (t\vec{v}_1 + \vec{v}_2) = 0.
\]

Since \( e^{\lambda t} \neq 0 \) and collecting the \( \vec{v}_1 \) and \( \vec{v}_2 \) terms, we have

\[
(c_1 + c_2 t) \vec{v}_1 + c_2 \vec{v}_2 = 0.
\]

Since \( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent, this gives \( c_1 + c_2 t = 0 \) and \( c_2 = 0 \). So clearly \( c_2 = 0 \), and substituting this into the first equation gives \( c_1 = 0 \) also. Hence \( \vec{x}_1(t), \vec{x}_2(t) \) are linearly independent for all \( t \).

(b) Let \( \vec{x}_1 \) and \( \vec{x}_2 \) be the two solutions from the previous subpart. Let \( W(t) = \det \begin{pmatrix} \vec{x}_1(t) & \vec{x}_2(t) \\ \vec{v}_1 & \vec{v}_2 \end{pmatrix} \) be the Wronskian of the two solutions. Compute \( W(t) \) and express your answer in terms of \( \lambda, t \) and \( W(0) \). [Hint: A direct computation under ‘exam pressure’ might not yield the answer in time. Here’s a hint at one shorter method – First find a \( 2 \times 2 \) matrix \( M \) (whose entries are functions of \( t \)) such that \( \begin{pmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{pmatrix} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix} M \).]

Solution. Let \( \vec{v}_1 = \begin{pmatrix} a \\ b \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} c \\ d \end{pmatrix} \). Then we have \( \vec{x}_1(t) = e^{\lambda t} \begin{pmatrix} a \\ b \end{pmatrix} \) and \( \vec{x}_2(t) = e^{\lambda t} \begin{pmatrix} ta + c \\ tb + d \end{pmatrix} \). Thus

\[
W(t) = \det \begin{pmatrix} e^{\lambda t} a & e^{\lambda t} (ta + c) \\ e^{\lambda t} b & e^{\lambda t} (tb + d) \end{pmatrix} = e^{2\lambda t} \det \begin{pmatrix} a & ta + c \\ b & tb + d \end{pmatrix}
\]

We wish to express this in terms of \( W(0) \), which is

\[
W(0) = \det \begin{pmatrix} \vec{x}_1(0) & \vec{x}_2(0) \\ \vec{v}_1 & \vec{v}_2 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \\ \vec{v}_1 & \vec{v}_2 \end{pmatrix}.\]

Now we note that

\[
\begin{pmatrix} a & ta + c \\ b & tb + d \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \\ \vec{v}_1 & \vec{v}_2 \end{pmatrix}.
\]
so we obtain for the determinant

$$W(t) = e^{2\lambda t} \det \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix} \det \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$= e^{2\lambda t} W(0) \cdot 1 = e^{2\lambda t} W(0)$$