1. Solve the ODE $y' + x = \sin(x)$ given the initial condition $y(0) = 1$

Solution. Separate variables:

$$
\frac{dy}{dx} + x = \sin(x) \\
\Rightarrow \quad y \, dy = (\sin(x) - x) \, dx \\
\Rightarrow \quad \int y \, dy = \int_0^x (\sin(x') - x') \, dx' \\
\Rightarrow \quad \frac{y^2}{2} - \frac{1}{2} = -\cos x - \frac{x^2}{2} - (-\cos 0 - 0) \\
\Rightarrow \quad y = \sqrt{3 - 2\cos x - x^2}
$$

Note that I didn’t say $y = \pm \sqrt{\cdots}$ above! The reason we only chose the positive version is because $y(0) = 1 > 0$. Thus (by continuity), $y = +\sqrt{\cdots}$ as long as the solution is defined.

2. Find the general solution of the ODE $dx/dt + x \ln(x) \tan(t) = x \cos(t)$. [HINT: Make the substitution $y = \ln(x)$.

This problem also involves remembering what $\int \tan(t) \, dt$ is. If you don’t remember what that indefinite integral is, then let $F(t) = \int \tan(t) \, dt$, and write your answer in terms of $F$, and a bunch of integrals involving $F$. You will only receive partial credit in this case.]

Solution. If $y = \ln x$, then $dy/dt = \frac{1}{x} \frac{dx}{dt}$. Substituting gives

$$
\frac{dx}{dt} + x \ln x \tan t = x \cos t \\
\Rightarrow \quad \frac{dy}{dt} + y \tan t = \cos t
$$

which is linear. Our integrating factor is given by

$$
\mu(t) = e^{\int \tan t \, dt} = e^{\ln \sec t} = \sec t
$$

Thus

$$
y = \frac{1}{\mu(t)} \int \mu(t) \cos(t) \, dt \\
= \cos t \int \sec t \cos t \, dt = \cos t \int 1 \, dt \\
= (t + C) \cos t
$$

where $C$ is an undetermined constant. Writing this in terms of $x$, we have $x = e^{(t+C)} \cos t$.

3. Determine the interval of existence for the following ODE’s, with given initial conditions. [Note the question only asks you to determine the interval of existence. You are not asked to solve these ODE’s.]

(a) $\frac{dx}{dt} + x = \tan(t)$ with initial condition $x(\pi) = 53$.

Solution. The equation is linear, so the solution exists as long as the coefficients are continuous. We know that tan is continuous on the intervals $((2n-1)\frac{\pi}{2}, (2n+1)\frac{\pi}{2})$, where $n \in \mathbb{Z}$. The only one of these intervals that contains the initial data ($t = \pi$) is the interval $(\frac{\pi}{2}, \frac{3\pi}{2})$. Thus the interval of existence is $(\frac{\pi}{2}, \frac{3\pi}{2})$.

(b) $\frac{dx}{dt} = -\frac{1}{x^2-1}$ with initial condition $x(0) = 0$. 

4. Newton’s law of cooling states that the rate at which an object loses heat is proportional to the difference between the temperature of the object and the surroundings. Suppose the room temperature is $19^\circ C$, and your bowl of freshly cooked Ramen Noodles$^\text{TM}$ has a temperature of $100^\circ C$. If 30 minutes later, your bowl of Ramen Noodles$^\text{TM}$ has temperature $28^\circ C$ (i.e. inedible), how much longer should you wait until your bowl of Ramen Noodles$^\text{TM}$ becomes completely inedible? [I know some would argue that the Ramen Noodles$^\text{TM}$ was inedible to start with anyway, but humor me. If you do things correctly, you should get a ‘nice’ answer.]

**Solution.** Let $\theta$ be the temperature of the Ramen Noodles$^\text{TM}$. By Newton’s law,

$$\frac{d\theta}{dt} = -\alpha(\theta - 19)$$

where $\alpha$ is some constant. Solving this ODE (with initial conditions $\theta(0) = 100$) gives

$$\frac{d\theta}{\theta - 19} = -\alpha dt \implies \ln(\theta - 19) - \ln(100 - 19) = -\alpha t \implies \ln\left(\frac{81}{\theta - 19}\right) = \alpha t.$$ 

For $t = 30$, we know $\theta = 28$. Thus

$$30\alpha = \ln\left(\frac{81}{28 - 19}\right) \implies \alpha = \frac{1}{30}\ln 9.$$ 

Now suppose $\theta = 22$ at some time $t$. Using the equation for $\theta$, and the value of $\alpha$ above, we have

$$\ln\left(\frac{81}{22 - 19}\right) = \left(\frac{1}{30}\ln 9\right) t \implies t = 30\frac{\ln 27}{\ln 9} = 30\frac{3\ln 3}{2\ln 3} = 45$$

So 45 minutes after you initially cooked the Ramen Noodles$^\text{TM}$, it will be at a temperature of $22^\circ C$. Thus you need to wait 15 more minutes for your Ramen Noodles$^\text{TM}$ to become completely inedible. □

5. Let $a \geq 0$, and consider the ODE $\frac{dx}{dt} = 1 + \sqrt{x - t}$ with initial condition $x(0) = a$. Find a value of $a \geq 0$ for which the given ODE (with the given initial conditions) does not have a unique solution. Further, for this value of $a$, find two distinct solutions. [HINT: One solution you have to find by guesswork. For the other solution, make the substitution $y = \sqrt{x - t}$.]

**Solution.** Let $f(x,t) = 1 + \sqrt{x - t}$. Note that $\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x - t}}$. This is not continuous when $x = 0$, $t = 0$. Thus your guess for $a$ should be $a = 0$.

Now let’s find two solutions: First by guesswork – Putting $x = t$ gets rid of the icky square root term, so let’s try it: If $x(t) = t$, then

$$\frac{dx}{dt} = 1 \quad \text{and} \quad 1 + \sqrt{x - t} = 1$$

Thus $x(t) = t$ is a solution of the given ODE with $x(0) = 0 = a$.

For the other solution, let’s try the substitution in the hint: If $y = \sqrt{x - t}$, then

$$\frac{dy}{dt} = \frac{1}{2\sqrt{x - t}} \left(\frac{dx}{dt} - 1\right) = \frac{1}{2\sqrt{x - t}}\sqrt{x - t} = \frac{1}{2}.$$ 

Thus our ODE reduces to $\frac{dy}{dt} = \frac{1}{2} \implies y = \frac{t}{2} + C$. When $t = 0$, $x = 0$ and hence $y = 0$. This gives us $C = 0$. Writing this in terms of $x$ we have

$$y = \frac{t}{2} \implies \sqrt{x - t} = \frac{t}{2} \implies x = \frac{t^2}{4} + t$$

This is our second solution with initial data $x(0) = 0 = a$. □

6. This question outlines how to solve the second order ODE $\frac{d^2x}{dt^2} + x = 0$. The techniques used in class so far will not provide a solution to this ODE, so at present some ingenuity is required.
(a) Suppose \( x \) satisfies \( \frac{d^2 x}{dt^2} + x = 0 \) with initial conditions \( x(0) = a \) and \( \frac{dx}{dt} \big|_{t=0} = b \). Show that \( x^2 + (\frac{dx}{dt})^2 \) is constant in time, and hence conclude that \( x^2 + (\frac{dx}{dt})^2 = a^2 + b^2 \).

Solution. Let \( y = x^2 + (\frac{dx}{dt})^2 \). Then

\[
\frac{dy}{dt} = 2x \frac{dx}{dt} + 2 \frac{dx}{dt} \frac{d^2 x}{dt^2} = 2x \frac{dx}{dt} - 2 \frac{dx}{dt} x = 0
\]

Thus \( y \) must be constant in time (it’s derivative is always 0). Using the initial data we see that \( y(0) = a^2 + b^2 \). Since \( y \) is constant in time, this gives \( x(t)^2 + (\frac{dx}{dt})^2 = y(t) = a^2 + b^2 \) for all time \( t \).

(b) The previous subpart shows that \( x \) satisfies the first order ODE \( x^2 + (\frac{dx}{dt})^2 = a^2 + b^2 \). Solve this ODE and find \( x \) explicitly as a function of \( t \).

Solution. For convenience, let \( r^2 = a^2 + b^2 \). Then our ODE reduces to

\[
\frac{dx}{dt} = \sqrt{r^2 - x^2}
\]

\[
\Rightarrow \frac{dx}{\sqrt{r^2 - x^2}} = dt \Rightarrow \sin^{-1} \left( \frac{x}{r} \right) = t + C
\]

Now using the initial data, we have \( \sin^{-1} \left( \frac{a}{r} \right) = C \), and hence

\[
x = r \sin \left( t + \sin^{-1} \left( \frac{a}{r} \right) \right)
\]

Now you could leave your answer in this form (since it is technically correct). However, if you’re going for ‘pretty’, then we can use the trig identities to simply further:

\[
x = r \sin \left( t + \sin^{-1} \left( \frac{a}{r} \right) \right) = r \sin t \cos \sin^{-1} \left( \frac{a}{r} \right) + r \cos t \sin \sin^{-1} \left( \frac{a}{r} \right)
\]

\[
= r \sqrt{1 - \frac{a^2}{r^2}} \sin t + a \cos t
\]

\[
= b \sin t + a \cos t
\]