1. Find a fundamental set of solutions to the following system of differential equations:

\[ y' = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix} y \]

The characteristic polynomial is:

\[ \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 \]

The only eigenvalue is 4, and it has algebraic multiplicity 2.

We now find the eigenspace of 4:

\[ \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \]
After row reducing, we find the eigenvector:

\[ v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]

Since the eigenspace was only one dimensional, we now need to find a generalized eigenvector.

\[
\begin{pmatrix}
1 & -1 & 1 \\
1 & -1 & 1
\end{pmatrix}
\]

Row reducing, we find the generalized eigenvector:

\[ w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

(Note that there infinitely many choices we can make for this generalized eigenvector.)

Putting these together, we have the following fundamental set of solutions:

\[ y_1(t) = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[ y_2(t) = e^{4t} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \]

Notes on grading: There were 5 points for finding the eigenvalue, 5 points for determining the eigenvector, 5 points for determining the generalized eigenvector and 5 points for writing down a fundamental set of solutions or the general solution.

2. Sketch a phase portrait for each of the following differential equations. The eigenvectors and eigenvalues are given to you.

(a)

\[ y' = \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} y. \]

You may use without justification the fact that:

\[
\begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
Notes on grading: I gave 5 points for determining this was a saddle. I gave 5 points for correctly determining the directions. I was lenient for those of you who confused \( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) with \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).

(b) \[
\mathbf{y}' = \begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix} \mathbf{y}.
\]

You may use without justification the fact that:

\[
\begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ i \end{pmatrix} = i \begin{pmatrix} 2 \\ i \end{pmatrix}
\]

We have a solution given by:

\[
\mathbf{y}_1(t) = \Re \left( e^{it} \begin{pmatrix} 2 \\ i \end{pmatrix} \right)
= \cos(t) \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \sin(t) \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

Thus, the solution curves are ellipses. The ellipse that passes through the point (2,0) also passes through the point (0,1).

The direction is determined by finding:

\[
\begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \end{pmatrix}
\]
thus it rotates clockwise.

Notes on grading: I allocated 5 points for determining that this was an ellipse. I allocated 3 points for having a graph that indicated that the ellipse passed through points \((2c,0)\) and \((0,c)\). I allocated 2 points for correctly determining that it rotates clockwise.

3. Find a fundamental set of solutions to the following system of differential equations:

\[
y' = \begin{pmatrix} -2 & 0 & 0 \\ -3 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} y.
\]

You may use, without justification, that

\[
\begin{pmatrix} -2 & 0 & 0 \\ -3 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.
\]

First, we compute the characteristic polynomial:

\[
\lambda^3 - 2\lambda + 4 = (\lambda + 2)(\lambda^2 - 2\lambda + 2) = (\lambda + 2)(\lambda - (1 + i))(\lambda - (1 - i)).
\]

(This factoring was made easier because we are given that \(\lambda = -2\) is a root.)
We already have the eigenvector for $\lambda = -2$, given by :

$$ v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. $$

We now need to find an eigenvector for $1 + i$.

$$ \begin{pmatrix} -3 - i & 0 & 0 \\ -3 & -i & 1 \\ 1 & -1 & -i \end{pmatrix} $$

After row-reducing, we find the eigenvector :

$$ v_2 = \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} $$

From these two eigenvectors, we can extract three linearly independent solutions as follows :

$$ y_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} $$

$$ y_2(t) = \text{Re} \left( e^{(1+i)t} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \right) = e^t \begin{pmatrix} 0 \\ \sin(t) \\ \cos(t) \end{pmatrix} $$

$$ y_3(t) = \text{Im} \left( e^{(1+i)t} \begin{pmatrix} 0 \\ -i \\ 1 \end{pmatrix} \right) = e^t \begin{pmatrix} 0 \\ -\cos(t) \\ \sin(t) \end{pmatrix} $$

Notes on grading: There were 5 points for finding the solution corresponding to the eigenvalue $-2$, 5 points for finding the other eigenvalues, 5 points for finding the eigenvectors and 5 points for writing down a fundamental set of solutions or the general solution.

4. Let

$$ A = \begin{pmatrix} -4 & 10 & -6 \\ -5 & 20 & -13 \\ -8 & 34 & -22 \end{pmatrix} $$

Find a fundamental set of solutions for

$$ y' = Ay. $$
You may use, without justification, that:

\[(A + 2 \text{Id})^3 = 0 \quad \text{and} \quad (A + 2 \text{Id})^2 = \begin{pmatrix} 2 & -4 & 2 \\ 4 & -8 & 4 \\ 6 & -12 & 6 \end{pmatrix}\]

(Hint: use this given fact to determine the eigenvalue(s) of \(A\), together with its/their algebraic multiplicity.)

(There are multiple solutions to this problem. This one seems easiest to me.)

From the fact that \((A + 2 \text{Id})^3 = 0\), we can conclude that \(-2\) is the only eigenvalue of \(A\) and it has multiplicity 3.

The generalized eigenspace of \(-2\) is thus:

\[\tilde{E}_{-2} = \text{Nullspace}((A + 2 \text{Id})^3) = \mathbb{R}^3.\]

We choose a basis of \(\mathbb{R}^3\) by \(\{e_1, e_2, e_3\}\).

From this, we obtain the following 3 linearly independent solutions to the differential equation:

\[y_1 = e^{-2t} \left( e_1 + (A + 2 \text{Id})e_1 t + (A + 2 \text{Id})^2 e_1 \frac{t^2}{2} \right)\]

\[= e^{-2t} \begin{pmatrix} 1 - 2t + t^2 \\ t(-5 + 2t) \\ t(-8 + 3t) \end{pmatrix}\]

\[y_2 = e^{-2t} \left( e_2 + (A + 2 \text{Id})e_2 t + (A + 2 \text{Id})^2 e_2 \frac{t^2}{2} \right)\]

\[= e^{-2t} \begin{pmatrix} -2t(t-5) \\ 1 - 4t^2 + 22t \\ -2t(3t-17) \end{pmatrix}\]

\[y_3 = e^{-2t} \left( e_3 + (A + 2 \text{Id})e_3 t + (A + 2 \text{Id})^2 e_3 \frac{t^2}{2} \right)\]

\[= e^{-2t} \begin{pmatrix} t(t-6) \\ t(2t-13) \\ 1 + 3t^2 - 20t \end{pmatrix}\]

Notes on grading: 5 points were awarded for noting that \(-2\) is an eigenvalue of algebraic multiplicity 3 (in other words, that there are no other eigenvalues), 10 points were awarded for seeing a way to solve this problem, e.g. that one can use \(e_1, e_2, e_3\) to solve the problem
or for writing down a way to compute $e^{At}$. One could earn the last 5 points for writing down the correct fundamental matrix or fundamental set of solutions.

5. Solve the following initial value problem:

$$y' = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} y + e^{3t} \begin{pmatrix} 1 \\ t \end{pmatrix}, \quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

You may use, without justification, that

for $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$, \quad $e^{At} = \begin{pmatrix} e^{3t}(1+t) & te^{3t} \\ -te^{3t} & e^{3t}(1-t) \end{pmatrix}$

We look for a solution of the form:

$$y(t) = e^{At} v(t).$$

Computing, this means:

$$A e^{At} v(t) + e^{3t} \begin{pmatrix} 1 \\ t \end{pmatrix} A e^{At} v(t) + e^{At} v'(t).$$

Thus,

$$v'(t) = e^{-At} e^{3t} \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1-t & -t \\ t & 1+t \end{pmatrix} \begin{pmatrix} 1 \\ t \end{pmatrix}$$

$$= \begin{pmatrix} 1-t-t^2 \\ 2t+t^2 \end{pmatrix}$$

Integrating, we get for $v(t)$:

$$v(t) = \begin{pmatrix} t-t^2/2-t^3/3+c_1 \\ t^2+t^3/3+c_2 \end{pmatrix}$$

Recall that $y(t) = e^{At} v(t)$. Thus, at $t = 0$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

The solution to the initial value problem is then:

$$y(t) = e^{3t} \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix} \begin{pmatrix} t-t^2/2-t^3/3+1 \\ t^2+t^3/3 \end{pmatrix}$$
6. Consider the following system of two masses and three springs, sliding on a frictionless table-top.

Assume each mass is of 1 kg, and the spring constants are as indicated. Then, the positions of the two masses are governed by the following system of differential equations:

\[
\begin{align*}
x'' &= -3x + y \\
y'' &= -3y + x
\end{align*}
\]

where \(x\) and \(y\) denote the positions of the two masses relative to the equilibrium point of this system (which occurs at \(x = y = 0\); we take the positive direction to be to the right).

(a) Transform this system of second order equations into a first order system of equations.

Let \(p = x'\) and \(q = y'\). Then:

\[
\begin{align*}
p' &= -3x + y \\
x' &= p \\
q' &= x - 3y \\
y' &= q.
\end{align*}
\]

(The above formed a complete solution. To continue further :) In matrix form, this gives:

\[
\begin{pmatrix}
p' \\
x' \\
q' \\
y'
\end{pmatrix} =
\begin{pmatrix}
0 & -3 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -3 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
p \\
x \\
q \\
y
\end{pmatrix}
\]
From physics, we know that the total energy of a system is the kinetic energy plus the potential energy. In this case, (since the mass is 1 kg) this is:

\[ E = \frac{1}{2}(x')^2 + \frac{1}{2}(y')^2 + \frac{1}{2}(2x^2) + \frac{1}{2}(2y^2) + \frac{1}{2}(x - y)^2. \]

Show this energy is constant along solutions to the differential equation.

To see that \( E \) is constant along solutions, we will differentiate it as a function of \( t \) along a curve \( x(t), y(t) \) that satisfies the differential equation:

\[
\frac{d}{dt} E = \frac{1}{2} x'' x' + \frac{1}{2} y'' y' + 2xx' + 2yy' + (x - y)(x' - y')
\]
\[
= \frac{1}{2} x'(-3x + y) + \frac{1}{2} y(-3y + x) + 2xx' + 2yy' + xx' - xy' - xy' + yy'
\]
\[
= 0.
\]

Without calculating the eigenvalues of the matrix you found in (a), what can you say about the eigenvalues, given that this energy \( E \) is constant?

The eigenvalues are purely imaginary.

If one of the eigenvalues had a positive real part, we would have a corresponding solution \((p(t), x(t), q(t), y(t))\) going to infinity as \( t \to \infty \). This would force the energy \( E = p^2/2 + q^2/2 + x^2 + y^2 + (x - y)^2/2 \) to go to infinity. The energy is constant, however, so this cannot occur.

If one of the eigenvalues had a negative real part, the corresponding solution \((p(t), x(t), q(t), y(t))\) would go to zero as \( t \to \infty \). This would force the energy to go to zero. Since the energy is constant, this cannot occur either.

The only possibility is for the eigenvalues to be purely imaginary.

(I took off a point for claiming the orbits were periodic — we haven’t done enough to figure this out. I took off an extra point for using this to justify that the eigenvalues were pure imaginary. I took off two points for a correct statement without justification.)