1. Solve each of the following initial value problems.

(a) \[ y' = 2y^2 + ty^2, \quad y(0) = 3. \]

The equation is separable:

\[ y' = y^2(2 + t). \]

Thus

\[ \int \frac{1}{y^2} dy = \int (2 + t) dt \]

\[ \frac{-1}{y} = 2t + \frac{t^2}{2} + C \]

\[ y = \frac{-1}{2t + t^2/2 + C}. \]

For \( y(0) = 3 \), need \( C = -\frac{1}{3} \):

\[ y(t) = \frac{-1}{t^2/2 + 2t - \frac{1}{3}}. \]

(b) \[ y' = 2e^{-yt^2}, \quad y(1) = 0. \]

\[ e^y y' = 2t^2 \]

\[ \int e^y dy = \int 2t^2 dt \]

\[ e^y = \frac{2}{3} t^3 + C \]

\[ y = \ln\left(\frac{2}{3} t^3 + C\right). \]

To solve \( 0 = y(1) = \ln(2/3 + C) \) need \( C = \frac{1}{3} \). Thus,

\[ y(t) = \ln\left(\frac{2}{3} t^3 + \frac{1}{3}\right). \]

(c) \[ y' - \sin(t)y = \sin(t)(2 + e^{-\cos(t)}), \quad y(0) = -1. \]

(This is the method by integrating factor. We could have used variation of parameters.)
First, find the factor $u$ so that $u' = -\sin(t)u$. Thus, $u(t) = e^{\cos(t)}$.

Then,

$$(y e^{\cos(t)})' = 2 \sin(t) e^{\cos(t)} + \sin(t)$$

$$y e^{\cos(t)} = -2 e^{\cos(t)} - \cos(t) + C$$

$$y = -2 - \cos(t) e^{-\cos(t)} + C e^{-\cos(t)}.$$

The initial condition gives :

$$-1 = y(0) = -2 - e^{-1} + C e^{-1}.$$

Thus, $C = 1 + e$. Thus,

$$y(t) = -2 - \cos(t) e^{-\cos(t)} + (1 + e) e^{-\cos(t)}.$$

**Grading guidelines:** Each question was worth 10 points. Solving for the initial condition was worth 3. I was not very generous with partial credit because the questions were supposed to be very basic.
2. Consider a cylindrical water tank of constant cross-sectional area $A$. Water is pumped into the tank at the constant rate $k$. Unfortunately, there is a small hole in the bottom of the tank, so the water leaks out. This hole has cross-sectional area $a$.

By Toricelli’s law, there is a constant $\alpha$ so that the water flows out with speed $\alpha \sqrt{2gh}$, where $h$ is the current depth of the water in the tank (and $g$ is the acceleration due to gravity).

(a) Explain briefly that the depth of the water in the tank at any time satisfies the differential equation:

$$\frac{d}{dt} h = \frac{(k - \alpha a \sqrt{2gh})}{A}.$$

(b) Find the equilibrium depth of the water. Determine whether it is asymptotically stable or unstable.

(c) Explain, in English, what this means for the depth of the water in the leaky tank. (A sentence or two will suffice.)

(a) If $V$ is the volume of water in the tank, then $h = \frac{V}{A}$, so $h' = \frac{V'}{A}$. The rate of water flowing in is $k$, the rate of water flowing out is the speed $\alpha \sqrt{2gh}$ multiplied by the area of the hole $a$. Putting all together we have

$$h' = \frac{V'}{A} = \frac{k - \alpha \sqrt{2gh} \times a}{A}$$

as claimed.

(b) The equilibrium depth of the water $h_0$ is the value of $h$ for which the right-hand side of the equation in (a) vanishes, i.e.:

$$0 = k - \alpha \sqrt{2gh_0} a.$$

The solution of the latter equation is given by

$$h_0 = \frac{k^2}{2a^2 g}.$$ 

For $h > h_0$ we have $\frac{k - \alpha \sqrt{2gh \times a}}{A} < \frac{k - \alpha \sqrt{2gh_0 \times a}}{A} = 0$ and for $h < h_0$ we have $\frac{k - \alpha \sqrt{2gh \times a}}{A} > \frac{k - \alpha \sqrt{2gh_0 \times a}}{A} = 0$. Hence, a solution $h(t)$ is strictly increasing if $h < h_0$ and strictly decreasing if $h > h_0$ and approaches $h_0$ in both cases. Thus, $h_0$ is stable.

(c) The result in (b) means that for any initial height $h \neq h_0$ the water level in the tank will tend to $h_0$ for $t \to \infty$. If $h = h_0$, then the water level will stay constant.

**Grading policy:** There were 5 points for part (a), 10 points for part (b) with 7 points for the computation of $h_0$ and 3 points for the stability and 5 points for part (c).
3. Find two different solutions $y_1(t)$ and $y_2(t)$ to the following differential equation that satisfy $y_1(0) = y_2(0)$:

$$y' = 5y^2,$$

By plugging it in it can be directly checked that $y_1(t) = 0$ is a solution of the equation. Further, it holds $y_1(0) = 0$. To find a second solution we may separate the variables and integrate to get

$$\int \frac{1}{5} y^{-2/5} dy = \int dt.$$

Computing the integrals we get

$$\frac{1}{3} y^{3/5} = t + C$$

with a constant $C \in \mathbb{R}$. Solving the latter equation for $y$ we end up with $y(t) = (3t + 3C)^{5/3}$. Letting $C = 0$ gives us a solution $y_2(t) = (3t)^{5/3}$ with $y_2(0) = 0$.

**Grading policy:** The solution $y(t) = (3t + 3C)^{5/3}$ was worth 7 points and the solution $y(t) = 0$ was worth 3 points.
4. (a) Find the general solution to the following system of differential equations

\[ \begin{align*}
x' &= 6x - 4y \\
y' &= 10x - 8y
\end{align*} \]

The equation can be written as:

\[ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ 10 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \]

The characteristic polynomial is \((\lambda - 2)(\lambda + 4)\).
To find an eigenvector for \(\lambda = 2\), need to solve the following augmented matrix:

\[
\begin{pmatrix}
4 & -4 & \mid & 0 \\
10 & -10 & \mid & 0
\end{pmatrix}
\]

Obtain \(v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\).

Similarly for \(\lambda = -4\), we obtain : \(v_2 = \begin{pmatrix} 2 \\ 5 \end{pmatrix}\).

The general solution is then :

\[ y(t) = C_1 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-4t} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \].

(b) Find the solution that satisfies the initial condition:

\[ x(0) = 3, \quad y(0) = 1. \]

Need to find \(C_1\) and \(C_2\) so that :

\[ \begin{align*}
3 &= C_1 + 2C_2 \\
1 &= C_1 + 5C_2
\end{align*} \]

Thus, \(3C_2 = -2\), so \(C_2 = -\frac{2}{3}\), and thus \(C_1 = 1 + \frac{10}{3} = \frac{13}{3}\).

\[ y(t) = \frac{13}{3} e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{2}{3} e^{-4t} \begin{pmatrix} 2 \\ 5 \end{pmatrix} \]

**Grading guidelines**: In (a), I allocated 4 marks for finding the eigenvalues, 4 marks for each of the two eigenvectors and 3 for the correct form of the general solution. For (b), I allocated 5 points. I took off one point for a minor algebra error, and two points if the algebra error was more serious.
Problem 5 Consider the autonomous differential equation
\[ y' = y (1 - y) (y - 2) \]  
(1)
(You may find it helpful to sketch a phase line to answer these questions.)

a) Find the equilibrium solutions of this equation. Classify them as asymptotically stable or unstable.

b) Suppose \( y_1(t) \) is the solution to the initial value problem
\[ y' = y (1 - y) (y - 2), \quad y(0) = 1/2. \]

i. What is its interval of existence?

ii. If \((\alpha, \beta)\) is the interval of existence you found above, what is \( \lim_{t \to \beta} y_1(t) \) ?

c) Suppose \( y_2(t) \) is the solution of the initial value problem
\[ y' = y (1 - y) (y - 2), \quad y(0) = 3. \]

Can you say anything about its interval of existence?

Solution and grading guidelines:

a) To find the equilibrium solutions of (1) we need to find the zeros of \( f(y) \equiv y (1 - y) (y - 2) \), which are 0, 1, and 2. To determine their stability, we need to plot \( f \), or at least determine the sign of \( f(y) \) for different values of \( y \in \mathbb{R} \). Since \( f(y) \) is a product of 3 factors, it is enough to determine the sign of each of the factors separately and then obtain the sign of \( f(y) \) from that. To do it methodically we can compile the following table.

<table>
<thead>
<tr>
<th>( t )</th>
<th>sign of ( y )</th>
<th>sign of ( 1 - y )</th>
<th>sign of ( y - 2 )</th>
<th>sign of ( f(y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t &lt; 0 )</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( 0 &lt; t &lt; 1 )</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( 1 &lt; t &lt; 2 )</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( 2 &lt; t )</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 1.

So the phase line has arrows pointing to the right on \((-\infty, 0)\) and \((1, 2)\); and arrows pointing to the left on \((0, 1)\) and \((2, +\infty)\). It follows that equilibria 0 and 2 are asymptotically stable, since there \( f(y) \) changes sign from + to - (equivalently "arrows point towards them"), while the equilibrium 1 is unstable ("arrows point away from 1").

There was a total of 10 points for part (a):

1. Finding the equilibria correctly was worth 3 points.

2. Determining the stability type was worth 7 points, with deductions depending on how involved the mistake was (misreading of a correct phase plane vs. a wrong answer with no explanation).

b) For this question having some sort of a phase line plot in (a) is almost indispensable:

i. The solution is sandwiched between 0 and 1, i.e. \( 0 < y_1(t) < 1 \) for all times \( t \in \mathbb{R} \). Hence by uniqueness the I.O.E. = \((-\infty, \infty)\).

Correct I.O.E. earned you 2 points; correct and comprehensible explanation earned you another 2.

ii. Since \( \beta = \infty \), we have \( \lim_{t \to \infty} y_1(t) = 0 \).

Correct limit earned you 4 points. If you provided no explanation of how you arrived at that (e.g. either no phase line, or no explanation in (i)), 2 points were deducted.

Additionally, anything suggesting that you think the I.O.E. is an interval on the \( y \)-axis was penalized by a deduction between 1 and 4 points. The total for part (b) was 8 points.
c) Since we know that \( y_2(0) = 3 \), the solution is decreasing (negative derivative) for all \( t \) and bounded below by the equilibrium \( y = 2 \), it follows that the solution definitely exists on the interval \( [0, \infty) \). So the I.O.E. is \((\alpha, \infty)\) for some unknown \( \alpha < 0 \).

Stating that the interval had the form \((\alpha, \infty)\) got you 1 point. Pointing out that \( \alpha < 0 \) was worth 1, too, making it a total of 2 points for part (c).

**Bonus problem.** Instead of considering a differential equation, we will consider an inequality instead:

\[
y' \leq C (1 + y), \quad \text{where} \quad C > 0 \text{ is a constant.}
\]

Suppose additionally that \( y \geq 0 \) for all \( t \).

a) Show that \( y(t) \leq (1 + y(0)) e^{Ct} - 1 \) for \( t \geq 0 \).

b) Use this to show that the following initial value problem:

\[
y' = \frac{y^2}{1 + y^2}, \quad y(0) = 1
\]

(2)

has an interval of existence \((-\infty, \infty)\).

**Solution and grading guidelines:**

a) The main idea is to use integration to get rid of the derivative. To preserve the inequality, however, a definite integral has to be used. In more detail: first, let us transform the inequality as follows

\[
y' \leq C (1 + y)
\]

\[
y' - Cy \leq C
\]

(3)

\[
(e^{-Ct}y)' \leq Ce^{-Ct}
\]

Now, we know that \( f(t) \leq g(t) \) for \( t \geq 0 \) implies that \( \int_0^t f(s) \, ds \leq \int_0^t g(s) \, ds \) for \( t \geq 0 \) (by comparing the areas underneath the graphs of \( f \) and \( g \)). Thus integrating the last line of (3) between 0 and \( t \) we get

\[
e^{-Ct} y(t) - y(0) \leq -e^{-Ct} + 1,
\]

which leads directly to the desired result.

There was a total of 6 points for this section. Using indefinite integrals instead of definite ones led to a 2 point deduction; claiming something to the effect of \( e^{-Ct} y(t) \leq -e^{-Ct} + A \) being true for ANY constant \( A \in \mathbb{R} \) led to additional deductions of 1-2 points. Answers with very little detail, or no clear direction earned only 0-2 points.

b) We are going to use a similar idea as in part (a). Let \( \varphi \) be a solution to the I.V.P. (2). Clearly \( y_1(t) \equiv 0 \) is an equilibrium solution, so that \( \varphi(t) > 0 \) for all \( t \) by uniqueness. Since \( \varphi(0) = 1 \) and \( \varphi \) is an increasing function on \((-\infty, 0)\) (\( \varphi'(t) > 0 \) for \( t < 0 \)), it automatically follows that \( \varphi \) exists on \((-\infty, 0]\).

To show that the solution also exists at all positive times, notice that

\[
\varphi'(t) = \frac{\varphi(t)^2}{1 + \varphi(t)^2} \leq \frac{1 + \varphi(t)^2}{1 + \varphi(t)^2} = 1 \quad \text{for all} \quad t.
\]

(4)

Integrating the outermost terms of this equality between 0 and \( t \geq 0 \) leads to

\[
\varphi(t) - \varphi(0) = \int_0^t \varphi(s) \, ds \leq \int_0^t ds = t,
\]

so that \( \varphi(t) \leq t + \varphi(0) \) for \( t \geq 0 \). Thus I.O.E. = \((-\infty, \infty)\).
A clear and correct answer was worth 4 points. Using indefinite integrals meant a deduction of 1-2 points. Doing something that looked like applying part (a) with no justification of why it applies, i.e. without showing that for some $C > 0$

$$\frac{y^2}{1 + y^2} \leq C(1 + y)$$

gained no points.