DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

INSTRUCTIONS:

- Your signature above indicates that you have abided by the Stanford Honor Code while writing this test.
- There are five questions, and a bonus. The exam will be counted out of 100 (though there are 110 possible points).
- You may quote theorems from your textbook or from class if you make an appropriate reference.
- Show all your work.
- No electronic devices of any kind (e.g. calculators, computers, cellphones) are allowed.

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<th>Question</th>
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1. Solve each of the following initial value problems.

(a) \[
y' = 2y^2 + ty^2, \quad y(0) = 3.
\]

(b) \[
y' = 2e^{-y}t^2, \quad y(1) = 0.
\]

(c) \[
y' - \sin(t)y = \sin(t)(2 + e^{-\cos(t)}), \quad y(0) = -1.
\]
This page has been left blank for your work.
2. Consider a cylindrical water tank of constant cross-sectional area $A$. Water is pumped into the tank at the constant rate $k$. Unfortunately, there is a small hole in the bottom of the tank, so the water leaks out. This hole has cross-sectional area $a$.

By Toricelli’s law, there is a constant $\alpha$ so that the water flows out with speed $\alpha \sqrt{2gh}$, where $h$ is the current depth of the water in the tank (and $g$ is the acceleration due to gravity).

(a) Explain briefly that the depth of the water in the tank at any time satisfies the differential equation:

$$\frac{dh}{dt} = \frac{k - \alpha a \sqrt{2gh}}{A}.$$

(b) Find the equilibrium depth of the water. Determine whether it is asymptotically stable or unstable.

(c) Explain, in English, what this means for the depth of the water in the leaky tank. (A sentence or two will suffice.)
3. Find two different solutions $y_1(t)$ and $y_2(t)$ to the following differential equation that satisfy $y_1(0) = y_2(0)$:

$$y' = 5y^2,$$
4. (a) Find the general solution to the following system of differential equations

\[ x' = 6x - 4y \]
\[ y' = 10x - 8y \]

(b) Find the solution that satisfies the initial condition:

\[ x(0) = 3, \quad y(0) = 1. \]
5. Consider the autonomous differential equation:

\[ y' = y(1 - y)(y - 2) \]

(You may find it helpful to sketch a phase line to answer these questions.)

(a) Find the equilibrium solutions of this equation. Classify them as asymptotically stable or unstable.

(b) Suppose \( y_1(t) \) is the solution to the initial value problem

\[ y' = y(1 - y)(y - 2), \quad y(0) = \frac{1}{2}. \]

i. What is its interval of existence?

ii. If \((\alpha, \beta)\) is the interval of existence you found above, what is \( \lim_{t \to \beta^-} y_1(t) \)?

(c) Suppose \( y_2(t) \) is the solution to the initial value problem

\[ y' = y(1 - y)(y - 2), \quad y(0) = 3. \]

Can you say anything about its interval of existence without solving the equation?
(Bonus) Instead of considering a differential equation, we will consider an inequality instead:

\[ y' \leq C(1 + y), \text{ where } C > 0 \text{ is a constant.} \]

Suppose additionally that \( y \geq 0 \) for all \( t \).

(a) Show that \( y(t) \leq (1 + y(0)) e^{Ct} - 1 \).

(b) Use this to show that the following initial value problem:

\[ y' = \frac{y^2}{1 + y^2}, \quad y(0) = 1 \]

has an interval of existence \((-\infty, \infty)\).
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